Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 7.11.2014 in the exercise class José Ramón Madrid Padilla (monitor)

Exercise 1.
Let $q = p^a$ be a prime power and $f \in \mathbb{F}_q[T]$ an irreducible polynomial of degree d. Show that the residue field of the f-adic valuation $v_f : \mathbb{F}_q(t) \to \mathbb{R} \cup \{\infty\}$ is isomorphic to $\mathbb{F}_{q^d}$. What is the valuation associated to $|\cdot|_\infty : \mathbb{F}_q(T) \to \mathbb{R}_{\geq 0}$ and what is its residue field?

Remark: For a non-trivial valuation v of $\mathbb{F}_q(T)$ with residue field k_v, we call $d = [k_v : \mathbb{F}_q]$ the *degree of the* v.

Exercise 2.
Let $v_T : \mathbb{F}_p(T) \to \mathbb{R} \cup \{\infty\}$ be the T-adic valuation. Show that the completion of O_v is the *ring of formal power series over* $\mathbb{F}_p$ *in* T

$$\mathbb{F}_p[[T]] \;=\; \left\{ \sum_{i=0}^{\infty} c_i T^i \;\middle|\; c_i \in \mathbb{F}_p \right\}$$

and that the completion of $\mathbb{F}_p(T)$ w.r.t. v_T is isomorphic to the *field of Laurent series*

$$\mathbb{F}_p((T)) \;=\; \left\{ \sum_{i=m}^{\infty} c_i T^i \;\middle|\; m \in \mathbb{Z}, c_i \in \mathbb{F}_p \right\}.$$

In particular, show that $\mathbb{F}_p((T))$ is the quotient field of $\mathbb{F}_p[[T]]$. Is the map

$$\begin{array}{cccc} \Phi : & \mathbb{F}_p[[T]] & \longrightarrow & \mathbb{Z}_p \\ & \sum_{n\geq 0} c_i T^i & \longmapsto & \sum_{n \geq 0} c_i p^i \end{array}$$

a bijection or even an isomorphism of rings if we consider $c_i \in \{0, \ldots, p-1\}$ as representatives of the respective residue fields?

Exercise 3.
Let K be a complete field w.r.t. an absolute value $|\cdot|$ and V a finite-dimensional K-vector space with a norm $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ such that $||av|| = |a| \cdot ||v||$ for all $a \in K$ and $v \in V$. Show that V is complete w.r.t. $||\cdot||$.

Hint: Let $e_1, \ldots, e_n$ be a basis for V. Show that $||\cdot||$ is equivalent to the maximum norm $\sum a_i e_i \mapsto \max\{|a_i|\}$, and that V is complete w.r.t. the maximum norm.

Exercise 4.
Prove the universal property of inverse limits (including the topology), i.e. Proposition 1 from section 5.4 of the lecture.