Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 14.11.2014 in the exercise class José Ramón Madrid Padilla (monitor)

Exercise 1.
Show that

1. the 2-adic absolute value $|\cdot|_2 : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ has precisely one extension $|\cdot|$ to $\mathbb{Q}(i)$ with residue field $\mathbb{F}_2$ such that $|\mathbb{Q}|_2$ is a subgroup of $|\mathbb{Q}(i)|$ of order 2;
2. the 3-adic absolute value $|\cdot|_3 : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ has precisely two extensions $|\cdot|$ and $|\cdot|'$ to $\mathbb{Q}(i)$, both with residue field $\mathbb{F}_2$ and value group $|\mathbb{Q}(i)| = |\mathbb{Q}(i)|' = |\mathbb{Q}|_3$;
3. the 5-adic absolute value $|\cdot|_5 : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ has precisely one extension $|\cdot|$ to $\mathbb{Q}(i)$ with residue field $\mathbb{F}_4$ and value group $|\mathbb{Q}(i)| = |\mathbb{Q}|_3$.

Hint: You can use without proof that $\mathbb{Z}[i]$ is a Euclidean domain w.r.t. the Euclidean function $f(a+ib) = \sqrt{a^2+b^2}$. This implies that $\mathbb{Z}[i]$ is a principal ideal domain and a factorial ring. Therefore the extensions of $|\cdot|_2$, $|\cdot|_3$ and $|\cdot|_5$ to $\mathbb{Q}(i)$ can be studied in terms of the prime factorizations of 2, 3 and 5 in $\mathbb{Z}[i]$.

Exercise 2.
Show that the completion of $\mathbb{Q}(i)$ w.r.t. to the extensions of $|\cdot|_2$ and $|\cdot|_5$ to $\mathbb{Q}(i)$ are quadratic field extension of $\mathbb{Q}_2$ and $\mathbb{Q}_5$, respectively. Show that the completion of $\mathbb{Q}(i)$ w.r.t. either extension of $|\cdot|_3$ to $\mathbb{Q}(i)$ is $\mathbb{Q}_3$.

Exercise 3.
Let v be a valuation of degree d of the rational function field $\mathbb{F}_q(T)$. Show that the completion of v is isomorphic to $\mathbb{F}_{q^d}((T))$ as a field.

Exercise 4.
Show that the equation $x^2+y^2=3$ has no solution in $\mathbb{Q}$, i.e. there exists no $(x,y) \in \mathbb{Q}$ such that $x^2+y^2=3$.

Hint: Show that the equation $x^2+y^2=3z^2$ has no solution in $\mathbb{Z}_2$ such that x, y, z are coprime. Use this to deduce that $x^2+y^2=3$ has no solution in $x, y \in \mathbb{Q}_2$, and thus not in $\mathbb{Q}$.

***Exercise 5** (Additional exercise for those who want to understand how local fields can be used to understand global fields)**.**
Let p be a prime number and $f \in \mathbb{Q}[T]$ be an irreducible polynomial of degree p. Assume that f has precisely $p-2$ roots in a completion L of $\mathbb{Q}$ w.r.t. to a non-trivial absolute value. Show that $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_p$ where K is the splitting field of f over $\mathbb{Q}$. Use this to determine the Galois group of the splitting field of $f = X^5 - 6X + 2$ over $\mathbb{Q}$.

Hint: Use Sylow's theorem to conclude that $\mathrm{Gal}(K/\mathbb{Q})$ contains a p-cycle, and use the assumption on the roots of f in L to show that $\mathrm{Gal}(K/\mathbb{Q})$ contains a 2-cycle.