Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 27.8.2014 in class José Ramón M. P. (monitor)

Exercise 1.
Let $L = K(a)$ where a is algebraic over K and L/K is of odd degree. Show that $L = K(a^2)$.

Exercise 2.
Proof "Fermat's small theorem": If K is a field of characteristic p, then $(a+b)^p = a^p + b^p$. Conclude that $\mathrm{Frob}_p : K \to K$ with $\mathrm{Frob}_p(a) = a^p$ is a field homomorphism.

[*Remark:* Frob_p is called the *Frobenius homomorphism in characteristic* p.]

Exercise 3.
Let $f = X^6 + X^3 + 1 \in \mathbb{Q}[T]$ and $L = \mathbb{Q}[T]/(f)$. Show that f is irreducible and find all field homomorphisms $L \to \mathbb{C}$.

[*Hint:* $f(X)$ divides $X^9 - 1$.]

Exercise 4.
Let L/E and E/K be field extensions with transcendental bases $S \subset L$ and $T \subset E$, respectively. Show that $S \cup T$ is a transcendental basis for L/K and conclude that the transcendental degree is additive for towers of field extensions $K_0 \subset K_1 \subset \cdots \subset K_n$.