Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 5.9.2014 in the exercise class José Ramón M. P. (monitor)

Exercise 1.
Which of the following polynomials is irreducible, which is separable?

1. $f(T) = T^p - 1$ in $\mathbb{F}_p[T]$.
2. $f(T) = T^p - x$ in $\mathbb{F}_p(x)[T]$ where x is a transcendental element over $\mathbb{F}_p$.

Exercise 2.
Let δ be a square root of $\overline{2} \in \mathbb{F}_3$ in the algebraic closure $\overline{\mathbb{F}_3}$ of $\mathbb{F}_3$. Show that $\mathbb{F}_3(\delta) = \{a + b\delta | a, b \in \mathbb{F}_3\}$ and that $\mathbb{F}_3(\delta)/\mathbb{F}_3$ is separable.

Exercise 3.
Let x be a transcendental element over K and $\mathrm{Aut}_K(K(x))$ the group of field isomorphisms $f : K(x) \to K(x)$ that fix every element of K. Let $\mathrm{GL}_2(K)$ be the group of invertible 2×2-matrices with coefficients in K and let $T = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \middle| a \in K^\times \right\}$ be the subgroup of central matrices. Show that $\mathrm{Aut}_K(K(x))$ is isomorphic to $\mathrm{GL}_2(K)/T$.

Exercise 4.
Let K be an algebraically closed field. Let $P(X,Y) \in K[X,Y]$ be an irreducible polynomial and $C = \{(x,y) \in K^2 | P(x,y) = 0\}$ the corresponding planar curve. Assume that there is a *rational map* $\varphi : K \to C$ *of degree* n, i.e. (1) $\varphi(t) = (a(t), b(t))$ with $a(T), b(T) \in K(T)$, which is defined for all but finitely many $t \in K$, and (2) for all but finitely many $(x,y) \in C$, the cardinality of $\varphi^{-1}(x,y)$ is n. Show with help of the theorem of Lüroth that C is a *rational curve*, i.e. there exists a rational map $\psi : K \to C$ of degree 1.