Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 12.9.2014 in the exercise class José Ramón Madrid Padilla (monitor)

Exercise 1.
Let L/K be a finite field extension and $p = \mathrm{char} K > 0$. Show that the inseparable degree $[L : K]_i$ is a power of p. Show further that for a tower of extensions $L/E/K$, we have $[L : K]_i = [L : E]_i : [E : K]_i$.

Exercise 2.
Find the splitting field L of $f = T^3 - 2 \in \mathbb{Q}[T]$. Which degree has $L/\mathbb{Q}$? Is $L = K[\sqrt[3]{2}]$?

Exercise 3.
Find all intermediate extensions of $\mathbb{Q}[\sqrt{2}, i]/\mathbb{Q}$ where i is a square root of -1.

Exercise 4.

1. Consider the purely transcendental extension $K = \mathbb{F}_3(x)/\mathbb{F}_3$ of transcendence degree 1, and let $\overline{K}$ be an algebraic closure of K. Let $a \in \overline{K}$ be a root of $f = T^3 - x$ and $b \in \overline{K}$ a root of $g = T^2 - 2$. Find the separable closure E of K in $K(a, b)$. What are the degrees $[K(a, b) : E]$ and $[E : K]$? What are the corresponding separable degrees and inseparable degrees?

2. Consider the purely transcendental extension $K = \mathbb{F}_3(x, y)/\mathbb{F}_3$ of transcendence degree 2, and let $\overline{K}$ be an algebraic closure of K. Let $a \in \overline{K}$ be a root of $f = T^3 - x$ and $b \in \overline{K}$ a root of $g = T^3 - y$. Show that every element c of $K(a, b)$ generates an extension $K(c)/K$ of degree at most 3. Conclude that $K(a, b)/K$ has infinitely many different intermediate fields, and that $K(a, b)$ cannot be generated by one element over K.