Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
Series 6 Oliver Lorscheid (professor)
To hand in at 3.10.2014 in the exercise class José Ramón Madrid Padilla (monitor)

Exercise 1.
Let ζ_{12} be a primitive 12-th root of unity. What is $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})$? Find primitive elements for all subfields E of $\mathbb{Q}(\zeta_{12})$.

Exercise 2.
Show that there is an n_i for $i = 1, 2, 3$ such that the following fields E_i are contained in $\mathbb{Q}(\zeta_{n_i})$. What are the smallest values for n_i?

1. $E_1 = \mathbb{Q}(\sqrt{2})$;
2. $E_2 = \mathbb{Q}(\sqrt{3})$;
3. $E_3 = \mathbb{Q}(\sqrt{-2})$;

Exercise 3.
For $n \geq 1$, let $\mu_n = \{\zeta \in \overline{\mathbb{Q}} | \zeta^n = 1\}$. For a positive divisor d of n, define

$$f_d = \prod_{\substack{\zeta \in \mu_n \\ \text{of order } d}} (T - \zeta).$$

1. Show that $\prod_{d|n} f_d = T^n - 1$.
2. Show that f_d has integral coefficients, i.e. $f_d \in \mathbb{Z}[T]$.
3. Let $\zeta \in \mu_n$ be of order d. Show that f_d is the minimal polynomial of ζ over $\mathbb{Q}$.
4. Conclude that $\deg f_d = \varphi(d)$ and that f_d is irreducible in $\mathbb{Z}[T]$.
5. Show that $f_d = T^{d-1} + \cdots + T + 1$ if d is prime.
6. Calculate f_d for $d = 1, \ldots, 12$.

The polynomial f_d is called the *d-th cyclotomic polynomial*.

Exercise 4. Let L be the splitting field of $T^3 - 2$ over $\mathbb{Q}$. Show that $\sqrt[3]{2}$, $\sqrt{-3}$ and ζ_3 are elements of L. Calculate $\mathrm{N}_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$ for $a = \sqrt[3]{2}$, $a = \sqrt{-3}$ and $a = \zeta_3$. Calculate $\mathrm{N}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$.