Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 10.10.2014 in the exercise class José Ramón Madrid Padilla (monitor)

Exercise 1.
Let

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of groups. Show that N and Q are solvable if and only if G is solvable.

Exercise 2.
Let L be the splitting field of a cubic polynomial f over K. Show that there is a subfield E of L such that $K \subset E \subset L$ is a tower of elementary radical extensions. What are E and L if $K = \mathbb{Q}$ and $f = T^3 - b \in \mathbb{Q}[T]$? When is E/K or L/E an Artin-Schreier extension?

Exercise 3.
Show that there is a radical extension L/K such that the normal closure L^{norm} of L over K admits no tower $K = K_0 \subset \cdots \subset K_r = L$ of elementary radical extensions.
Hint: Conclude from the previous exercise that the splitting field of a polynomial $f = T^3 - b$ has even degree over $\mathbb{Q}$. Show that $\zeta_7 + \zeta_7^{-1}$ generates a cyclic extension L over $\mathbb{Q}$ of degree 3. Conclude that $L/\mathbb{Q}$ is an example with the desired properties.

Exercise 4. Let L/K be a Galois extension and let

$$\begin{aligned} M_a : \quad L &\longrightarrow L \\ b &\longmapsto a \cdot b \end{aligned}$$

be the K-linear map associated with an element $a \in L$. Show that the trace of M_a equals $\mathrm{Tr}_{L/K}(a)$ and that the norm of M_a equals $\mathrm{N}_{L/K}(a)$.
Hint: Use Exercise 1 from Series 1.