
Exercise 1.
Let

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of groups. Show that N and Q are solvable if and only if G is solvable.

Exercise 2.
Let ζ_{12} be a primitive 12-th root of unity. What is $\mathrm{Gal}(\mathbb{Q}(\zeta_{12}/\mathbb{Q})$? Find primitive elements for all subfields E of $\mathbb{Q}(\zeta_{12})$.

Exercise 3.
Show that there is an n_i for $i = 1, 2, 3$ such that the following fields E_i are contained in $\mathbb{Q}(\zeta_{n_i})$. What are the smallest values for n_i?

1. $E_1 = \mathbb{Q}(\sqrt{2})$;
2. $E_2 = \mathbb{Q}(\sqrt{3})$;
3. $E_3 = \mathbb{Q}(\sqrt{-3})$;

Hint: Try to realize $\sqrt{2}$ and $\sqrt{3}$ as the side length of certain right triangles. Which angles do occur?

Exercise 4 (Cyclotomic polynomials)**.**
Let $\mu_\infty = \{\zeta \in \overline{\mathbb{Q}} | \zeta^n = 1 \text{ for some } n \geq 1\}$. Define

$$f_d = \prod_{\substack{\zeta \in \mu_\infty \\ \text{of order } d}} (T - \zeta).$$

1. Show that $\prod_{d|n} f_d = T^n - 1$ for $n \geq 1$.
2. Show that f_d has integral coefficients, i.e. $f_d \in \mathbb{Z}[T]$.
3. Let $\zeta \in \mu_\infty$ be of order d. Show that f_d is the minimal polynomial of ζ over $\mathbb{Q}$.
4. Conclude that $\deg f_d = \varphi(d)$ and that f_d is irreducible in $\mathbb{Z}[T]$.
5. Show that $f_d = T^{d-1} + \cdots + T + 1$ if d is prime.
6. Calculate f_d for $d = 1, \ldots, 12$.

The polynomial f_d is called the *d-th cyclotomic polynomial.*