Exercises for Algebra II
Series 7
To hand in until 21.9.

Instituto Nacional de Matemática Pura e Aplicada
Carolina Araujo and Oliver Lorscheid
Roberto Alvarenga Jr. (monitor)

Exercise 1.
Let K be a field and L the splitting field of a cubic polynomial f over K. Assume that L/K is separable. Show that there is a subfield E of L such that $K \subset E \subset L$ is a tower of elementary radical extensions (with possibly $L = E$ or $E = K$). In which situations are E/K and L/E cyclotomic, Kummer and Artin-Schreier? What are E and L if $K = \mathbb{Q}$ and $f = T^3 - b \in \mathbb{Q}[T]$?

Exercise 2.
Show that there is a solvable extension L/K that not radical.

Hint: Conclude from the previous exercise that the splitting field of a polynomial $f = T^3 - b$ has even degree over $\mathbb{Q}$ if it is not equal to $\mathbb{Q}$. Show that $\zeta_7 + \zeta_7^{-1}$ generates a cyclic extension L over $\mathbb{Q}$ of degree 3. Conclude that $L/\mathbb{Q}$ is an example with the desired properties.

Exercise 3.
Which roots of the following polynomials are construcible over $\mathbb{Q}$?

1. $f_1 = T^4 - 2$
2. $f_2 = T^4 - T$
3. $f_3 = T^4 - 2T$

Exercise 4.
Let K be a subfield of $\mathbb{C}$ and a a root of $T^2 - b \in K[T]$. Show that every element of $K(a)$ is constructible over K. Use this to explain the relationship between the two definitions of constructible numbers from sections 1.1 and 4.7 of the lecture.

*__Exercise 5.__ [1]
Let $\mathbb{F}_p[x, y]$ be the polynomial ring in two variables x and y and $\mathbb{F}_p(x, y)$ its fraction field. Let $\sqrt[p]{x}$ be a root of $T^p - x$ and $\sqrt[p]{y}$ be a root of $T^p - y$.

1. Show that $\mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$ is a field extension of $\mathbb{F}_p(x, y)$ of degree p^2.
2. Show that $a^p \in \mathbb{F}_p(x, y)$ for every $a \in \mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$.
3. Conclude that the field extension $\mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y}) \,/\, \mathbb{F}_p(x, y)$ has no primitive element and that it has infinitely many intermediate extensions.

---

[1] The starred exercises are not to hand in. But it is advised to work on these exercises, and possibly to discuss them in the exercise class.