Exercises for Algebraic Number Theory IMPA
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To hand in at 14.1.2015 in the exercise class Roberto Alvarenga Jr. (monitor)

Exercise 1.
Show the following properties of the norm $N : \mathbb{Z}[i] \to \mathbb{Z}$ for all $z, z' \in \mathbb{Z}[i]$ and $x \in \mathbb{Z}$.

1. $N(z) \geq 0$;
2. $N(zz') = N(z)N(z')$;
3. $N(x) = x^2$;
4. $N(z) = 1$ if and only if $z \in \mathbb{Z}[i]^\times$.

What is $\mathbb{Z}[i]^\times$?

Exercise 2.
Show that $\mathbb{Z}[i]$ equals the set of all $z \in \mathbb{Q}[i]$ for which there are $a, b \in \mathbb{Z}$ such that $z^2 + az + b = 0$. How is b related to the norm of z? Can you express a as a term in z?

Exercise 3.

1. Show that $\mathbb{Z}[\sqrt{2}]$ is a unique factorization domain.
2. Show that $1 + \sqrt{-5}$ is irreducible, but not prime, in $\mathbb{Z}[\sqrt{-5}]$. Is $\mathbb{Z}[\sqrt{-5}]$ a unique factorization domain?

Exercise 4.
Prove Euclid's theorem about Pythagorean triples: for every solution $(a, b, c) \in \mathbb{Z}^3_{>0}$ of $x^2 + y^2 = z^2$, there are unique $k, m, n \in \mathbb{Z}_{>0}$ with $(m, n) = 1$ such that $a = k \cdot (n^2 - m^2)$, $b = k \cdot 2nm$ and $c = k \cdot (n^2 + m^2)$ (up to a permutation of a and b).

*__Exercise 5.__ [1]

1. Show that every prime element of an integral domain is indecomplosable.
2. Show that in a unique factorization domain, every irreducible element is prime.
3. Show that every Euclidean ring is a principle ideal domain.
4. Show that every principle ideal domain is a unique factorization domain.

[1] The starred exercises are not to hand in. But it is advised to work on these exercises, and possibly to discuss them in the exercise class.