**Exercises for Algebraic Number Theory** Oliver Lorscheid
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To hand in until 22.1.2015 with Emilio Peixoto Assemany or Roberto Alvarenga Jr.

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**Exercise 1.**
Which of the following numbers are algebraic integers?

$$\sqrt{2}, \quad \frac{1}{2}, \quad \sqrt{\frac{1}{2}}, \quad \frac{1+\sqrt{5}}{2}, \quad \frac{3+2\sqrt{6}}{1-\sqrt{6}}.$$

**Exercise 2.**
Every unique factorization domain is integrally closed. Conclude that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

**Exercise 3.**
Let $L/K$ be a separable field extension of degree $n$ with basis $(1, a, \dots, a^{n-1})$ for some $a \in L$. Then $d(1, a, \dots, a^{n-1}) = \prod_{i<j}(a_i - a_j)^2$.

**Exercise 4.** Let $L$ be a number field and $\mathcal{O}_L$ its integers.

1. Let $M$ be a non-trivial finitely generated $\mathcal{O}_L$-submodule of $L$. Show that the discriminant $d(M) = d(b_1, \dots, b_n)$ of $M$ does not depend on the choice of a $\mathbb{Z}$-basis $(b_1, \dots, b_n)$ of $M$.
2. Let $M \subset M'$ be two non-trivial finitely generated $\mathcal{O}_L$-submodules of $L$. Show that the index $(M' : M)$ is finite and that $d(M) = (M' : M)^2 d(M')$.

**Exercise 5.**

1. Show that every quadratic extension of $\mathbb{Q}$ is of the form $\mathbb{Q}[\sqrt{D}]$ with $D \in \mathbb{Z}$.
2. Let $D \in \mathbb{Z}$ be squarefree and different from 0 and 1. Consider $L = \mathbb{Q}[\sqrt{D}]$ and its integers $\mathcal{O}_L$. Show that a basis of $\mathcal{O}_L$ over $\mathbb{Z}$ is given by

$$\begin{array}{ll} \{1, \sqrt{D}\} & \text{if } D \equiv 2 \text{ or } 3 \pmod 4, \\ \{1, (1+\sqrt{D})/2\} & \text{if } D \equiv 1 \pmod 4. \end{array}$$

3. Calculate the discriminant of $L$ in either case.

*Hint:* Use the norm and trace to calculate the minimal polynomial of an element $a \in L$.

*__Exercise 6.__ Verify all properties of norm and trace as claimed in Proposition 2 of the lecture. To keep the proofs simple, you may assume that all field extensions are separable.

*__Exercise 7__ (Chinese Remainder theorem)**.**
Let $A$ be a ring, $I_1, \ldots, I_n$ be ideals of $R$ with $I_i + I_j = A$ for $i \neq j$ and $J = \bigcap_{i=1}^n I_i$. Then there is a canonical isomorphism of rings

$$A/J \xrightarrow{\sim} \bigoplus_{i=1}^{n} A/I_i.$$

*__Exercise 8.__ Let $f : A \to B$ be a ring homomorphism and $I$ an ideal of $B$. Show that $f^{-1}(I)$ is an ideal of $A$, and show that if $I$ is a prime ideal, then so is $f^{-1}(I)$. Is the converse statement true?

*__Exercise 9.__ Let $A$ be a ring. Show that the $A$-submodules of $A$ are precisely the ideals of $A$.

*__Exercise 10.__ Let $A$ be a ring, $a, b \in A$ and $I, J \subset A$ ideals.

1. Show that $a \,|\, b$ if and only if $(a) \,|\, (b)$.
2. Show that $I \cap J \,|\, I \cdot J$. Is the converse true?
3. Show that $I \cap J$ is the least common multiple of $I$ and $J$.
4. Show that $I + J$ is the greatest common divisor of $I$ and $J$.

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The starred exercises are not to hand in.