
Exercise 1.
Show that the additive group of $\mathbb{Q}$ is a torsionfree $\mathbb{Z}$-module. Show that every free submodule of $\mathbb{Q}$ is cyclic, and show that the same is true for finitely generated submodules of $\mathbb{Q}$. Give an example of a proper submodule $N \subsetneq \mathbb{Q}$ that is not cyclic.

Exercise 2.
Let $A = \mathbb{C}[T]$ and consider $N = \mathbb{C}^n$ as an A-module by letting T act as a complex $n \times n$-matrix M. Show that N is a cyclic A-module if and only if the Jordan normal form of M consists of only one Jordan block, i.e. if M is conjugated to a matrix of the form

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{C}$.

Exercise 3.
Let A be an integral domain. Show that a homomorphism $f : M \to N$ of A-modules restricts to a homomorphism $T(M) \to T(N)$ between their respective torsion modules. Show that this defines a left exact functor $T : A - \mathrm{Mod} \to A - \mathrm{Mod}$.

Exercise 4. Show that the following properties for an A-module P are equivalent.

1. The functor $\mathrm{Hom}(-, P)$ is exact.
2. There is an A-module Q such that $P \oplus Q$ is free.
3. Every short exact sequence of A-modules of the form $0 \to N \to M \to P \to 0$ splits.
4. For every epimorphism $p : M \to Q$ of A-modules and every homomorphism $f : P \to Q$, there is a homomorphism $g : Q \to M$ such that $f = p \circ g$.

An A-module P with these properties is called *projective*. Conclude that every free A-module is projective. Show that $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module that is not free.

Remark: An A-module I is *injective* if and only if $\mathrm{Hom}(I, -)$ is exact. It can be shown that there are analogous characterizations as in (3) and (4) for injective modules. However, there is no explicit description as in (2) in general. For $A = \mathbb{Z}$, one can show that a $\mathbb{Z}$-module I is injective if and only if it is *divisible*, i.e. for every $m \in I$ and every $l > 0$ there exists an $n \in I$ such that $l.n = m$.

Exercise 5 (Bonus)**.**
Let $f : A \to B$ be a ring homomorphism, M an A-module and N a B-module. Show the following assertions.

1. The abelian group $f_*M = M \otimes_A B$ is a B-module with respect to the action $a.(m \otimes b) = m \otimes ab$. A homomorphism $M \to M'$ of A-modules induces a homomorphism $M \otimes_A B \to M' \otimes_A B$ of B-modules via $m \otimes b \mapsto f(m) \otimes b$. This defines a functor $f_* : A - \text{Mod} \to B - \text{Mod}$, called the *push forward along f*.

2. The abelian group $f^*N = N$ is an A-module with respect to the action $a.m = f(a).m$. A homomorphism $N \to N'$ of B-modules is also a homomorphism of A-modules. This defines a functor $f^* : B - \text{Mod} \to A - \text{Mod}$, called the *pull back along f*.

3. The association $(g : M \to f^*N) \mapsto (m \otimes b \mapsto b.g(m))$ defines a bijection $\Phi_{M,N} : \text{Hom}_A(M, f^*N) \to \text{Hom}_B(f_*M, N)$. This bijection is *functorial* in M and N, i.e. a homomorphism $h : M \to M'$ of A-modules yields a commutative square

$$\begin{array}{ccc} \text{Hom}_A(M', f^*N) & \xrightarrow{\Phi_{M',N}} & \text{Hom}_B(f_*M', N) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle (f_*(h))^*} \\ \text{Hom}_A(M, f^*N) & \xrightarrow{\Phi_{M,N}} & \text{Hom}_B(f_*M, N) \end{array}$$

and a homomorphism $h : N \to N'$ of B-modules yields a commutative square

$$\begin{array}{ccc} \text{Hom}_A(M, f^*N) & \xrightarrow{\Phi_{M,N}} & \text{Hom}_B(f_*M, N) \\ \downarrow{\scriptstyle (f^*(h))_*} & & \downarrow{\scriptstyle (h)_*} \\ \text{Hom}_A(M, f^*N') & \xrightarrow{\Phi_{M,N'}} & \text{Hom}_B(f_*M, N'). \end{array}$$

Remark: This exercise verifies that the functor $f_* : A - \text{Mod} \to B - \text{Mod}$ is *left adjoint* to $f^* : B - \text{Mod} \to A - \text{Mod}$.