
Exercise 1.
Consider the submodule N of the $\mathbb{Z}$-module $M = \mathbb{Z}^2$ that is generated by $(1,1)$ and $(1,-1)$. Determine the free rank and the elementary divisors of M/N. Determine further the ideals of $\mathbb{Z}$ that occur as the annihilator $\mathrm{Ann}_{\mathbb{Z}}(m)$ of an element m of M/N.

Exercise 2. Consider the $\mathbb{C}[T]$-module $M = \mathbb{C}^3$ where T acts as one of the matrices

$$(1)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad (2)\quad T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad (3)\quad T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$(4)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \qquad (5)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad (6)\quad T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

and where λ, μ and ν are pairwise distinct complex numbers. Determine in each case the characteristic polynomial and the minimal polynomial of T, as well as the elementary divisors and the invariant factors of M.

Exercise 3.
Let A be a ring and $\mathrm{Mat}_{n\times n}(A)$ the set of $n \times n$-matrices with coefficients in A.

1. Show that $\mathrm{Mat}_{n\times n}(A)$ is a noncommutative ring with respect to matrix addition and matrix multiplication. What are 0 and 1?

2. Show that the inclusion $f : A \to \mathrm{Mat}_{n\times n}(A)$ as diagonal matrices is a homomorphism of (noncommutative) rings, i.e. $f(a+b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(1) = 1$.

3. The *determinant* is the map $\det : \mathrm{Mat}_{n\times n}(A) \to A$ that sends a matrix $T = (a_{i,j})_{i,j=1,\dots,n}$ to the element

$$\det(T) \quad = \quad \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

   of A. Show that det is multiplicative, i.e. $\det(T \cdot T') = \det(T) \cdot \det(T')$ and $\det(1) = 1$.

4. Show that a matrix T is a unit in $\mathrm{Mat}_{n\times n}(A)$, i.e. $TT' = 1$ for some matrix T', if and only if $\det(T)$ is a unit in A.

Exercise 4.
An A-module M is *flat* if $- \otimes_A M$ is exact.

1. Show that every free A-module is flat. Conclude that every projective A-module is flat. (*Hint:* Use property 2 from Exercise 4 on List 10.)
2. Let I be an ideal of A. Show that $I \otimes_A M \simeq IM$ if M is flat.
3. Show that every ideal of a principal ideal domain A is flat as an A-module, and conclude that $IJ \simeq I \otimes_A J$ for all ideals I and J of A.

Exercise 5 (Bonus)**.**
An A-module is called *Noetherian* if every of its submodules is finitely generated over A. A ring A is called *Noetherian* if it is Noetherian as a module over itself, i.e. if all of its ideals are finitely generated.

1. Let A be any ring. Show that if $M \to Q$ is an epimorphism of A-modules and M is Noetherian, then Q is Noetherian.
2. Let $M_1, \ldots, M_n$ be Noetherian A-modules. Show (by induction on n) that $\bigoplus_{i=1}^n M_i$ is Noetherian.
3. Let A be Noetherian ring. Conclude that every finitely generated A-module is Noetherian.
4. Use this to give an alternative proof of Corollary 2 of section 4.8 of the lecture.