
Exercise 1.
Let N be the submodule of the free $\mathbb{Z}$-module $\mathbb{Z}^4$ that is generated by

$$(1,1,1,0), \quad (1,1,0,1), \quad (1,0,1,1), \text{ and } \quad (0,1,1,1).$$

Determine a basis $\{b_1,\dots,b_4\}$ of $\mathbb{Z}^4$ and integers $a_1,\dots,a_4$ such that $\{a_1b_1,\dots,a_4b_4\}$ is a basis of N.

Exercise 2.
Prove the theorem of the Smith normal form.

Exercise 3.
Let k be a field, M a finite dimensional k-vector space and $\varphi : M \to M$ a k-linear map. Let $I_1 = (f_1),\dots,I_s = (f_s)$ be the invariant factors of M as $k[T]$-module where T acts as φ and where $f_1,\dots,f_s$ are monic polynomials. Show that $\prod_{i=1}^s f_i$ is the characteristic polynomial of φ.

Hint: Reduce the situation to the case where M is cyclic and use that in this case, the characteristic polynomial equals the minimal polynomial.

Exercise 4.
Let k be a field and M a finite dimensional k-vector space. A k-linear map $\varphi : M \to M$ is called *diagonalizable* if it acts as a diagonal matrix with respect to some basis of M. Show that φ is diagonalizable if and only if the minimal polynomial is of the form

$$\min{}_\varphi = \prod_{i=1}^n (T - \alpha_i)$$

for pairwise distinct $\alpha_1,\dots,\alpha_n \in k$. Is the $\mathbb{C}$-linear map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ given by the matrix $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ for the standard basis of $\mathbb{C}^2$ diagonalizable?

Exercise 5 (Bonus)**.**
Let k be a field, M and N finite dimensional k-vector spaces, and $\varphi : M \to M$ and $\psi : N \to N$ k-linear maps. Assume that their respective characteristic polynomials factor as

$$\mathrm{char}_\varphi = \prod_{i=1}^m (T - \alpha_i), \text{ and } \quad \mathrm{char}_\psi = \prod_{j=1}^n (T - \beta_j).$$

Show that the formula $\varphi \otimes \psi(m \otimes n) = \varphi(m) \otimes \psi(n)$ defines a k-linear homomorphism $\varphi \otimes \psi : M \otimes_k N \to M \otimes_k N$, whose characteristic polynomial is

$$\mathrm{char}_{\varphi\otimes\psi} = \prod_{i,j} (T - \alpha_i\beta_j).$$