

Exercise 1.

Let M be a free A -module of rank $n > 0$ with basis $\{b_1, \dots, b_n\}$. Show that the map

$$\begin{array}{ccc} A[T_1, \dots, T_n] & \longrightarrow & \text{Sym}(M) \\ T_i & \longmapsto & b_i \end{array}$$

is a graded isomorphism of graded A -algebras. Show that the dimension of $T^l(M)$ is n^l and that the dimension of $\text{Sym}^l(M)$ is $\binom{n+l-1}{l}$.

Exercise 2.

Let M be a finitely generated A -module. The *exponential map* $\exp : \Lambda(M) \rightarrow \Lambda(M)$ is defined by the formula

$$\exp(x) = 1 + \sum_{k \geq 1} \frac{1}{k!} \underbrace{(x \wedge \dots \wedge x)}_{k\text{-times}}.$$

Show that $\exp(x)$ is equal to a finite sum, and therefore well-defined as an element of $\Lambda(M)$. Calculate the expressions $\exp(m)$ and $\exp(m \wedge n + o \wedge p)$ where $m, n, o, p \in M$. Does the formula $\exp(x + y) = \exp(x) \wedge \exp(y)$ hold for any $x, y \in \Lambda(M)$?

Exercise 3.

Let M be an A -module. Consider the ideals

$$I = \langle m \otimes m \mid m \in M \rangle \quad \text{and} \quad J = \langle m \otimes n + n \otimes m \mid m, n \in M \rangle$$

of $T(M)$. Show that $I = J$ if 2 is invertible in A . Give an example for A and M where $I \neq J$.

Exercise 4.

Let M be a free A -module with basis $\{b_1, \dots, b_n\}$. For $j = 1, \dots, l$, consider the elements

$$m_j = \sum_{i=1}^n a_{i,j} b_i$$

of M where $a_{i,j} \in A$. The element $m_1 \wedge \dots \wedge m_l \in \Lambda^l(M)$ can be uniquely written as

$$m_1 \wedge \dots \wedge m_l = \sum c_\sigma \cdot b_{\sigma(1)} \wedge \dots \wedge b_{\sigma(l)}$$

where σ ranges through all strictly order preserving maps $\sigma : \{1, \dots, l\} \rightarrow \{1, \dots, n\}$ (i.e. $\sigma(i) < \sigma(j)$ if $i < j$). Show that

$$c_\sigma = \det(a_{i,j})_{\substack{i \in \text{im} \sigma, \\ j=1, \dots, l}}$$

Exercise 5 (Bonus).

This is a continuation of Exercise 4. However, we assume that $A = k$ is field for this exercise. Consequently, M is a k -vector space.

1. Show that $m_1 \wedge \dots \wedge m_l \neq 0$ if and only if $\{m_1, \dots, m_l\}$ is linearly independent.
2. Assume that $\{m_1, \dots, m_l\}$ and $\{m'_1, \dots, m'_l\}$ are linearly independent subsets of M . Show that there is a $\lambda \in k^\times$ such that

$$m'_1 \wedge \dots \wedge m'_l = \lambda \cdot m_1 \wedge \dots \wedge m_l.$$

if and only if $\{m_1, \dots, m_l\}$ and $\{m'_1, \dots, m'_l\}$ span the same l -dimensional subvector space N of M .

Hint: If they span the same subvector space N , then one can find a $l \times l$ -base change matrix. What is the effect of this matrix on the coefficients c_σ from Exercise 4?

3. Define $\mathbb{P}(\Lambda^l(M)) = (\Lambda^l(M) - \{0\})/k^\times$ as the set of equivalent classes of nonzero elements of $\Lambda^l(M)$ modulo scalar multiplication by nonzero $\lambda \in k^\times$. Conclude from the previous part of the exercise that there is a well-defined inclusion

$$\{l\text{-dimensional subvector spaces of } M\} \longrightarrow \mathbb{P}(\Lambda^l(M)).$$

Remark: The set $\mathbb{P}(\Lambda^l(M))$ is called the *projective space of $\Lambda^l(M)$* , the above inclusion is called the *Plücker embedding* and its image is called the *Grassmann variety $\text{Gr}(l, n)$ of l -subspaces in n -space*.