Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
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Exercise 1.
Let $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ be a functor. Show that the following properties are equivalent.

1. Every object of $\mathcal{D}$ has an initial morphism $(\widetilde{Y}, \eta_Y : Y \to \mathcal{G}(\widetilde{Y})$ to $\mathcal{G}$.
2. $\mathcal{G}$ has a left adjoint $\mathcal{F} : \mathcal{D} \to \mathcal{C}$.
3. $\mathcal{G}$ is part of a counit-unit adjunction $(\epsilon, \eta) : \mathcal{F} \dashv \mathcal{G}$ between $\mathcal{C}$ and $\mathcal{D}$.

Exercise 2.
Show that right (left) adjoints are unique up to unique natural isomorphism. This means that if $\mathcal{G}$ and $\mathcal{G}'$ are right adjoints of a functor $\mathcal{F} : \mathcal{D} \to \mathcal{C}$, then there is a unique natural isomorphism $\eta : \mathcal{G} \to \mathcal{G}'$; and similar for left adjoints.

Exercise 3.
Every object A of a category $\mathcal{C}$ defines the functor $h^A = \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathrm{Set}$ that sends an object B to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and a morphism $f : B \to B'$ to the map $f_* : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, B')$, defined by $f_*(g) = f \circ g$.

Verify that every morphism $f : A' \to A$ defines a natural transformation $\eta_f : h^A \to h^{A'}$ that sends $g : A \to B$ in $h^A(B)$ to $g \circ f : A' \to B$ in $h^{A'}(B)$. Show that this defines a bijection between $\mathrm{Hom}_{\mathcal{C}}(A', A)$ and the set of all natural transformations $\eta : h^A \to h^{A'}$. Conclude that if there is a family of bijections

$$\big\{\Phi_B : \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A', B)\big\}_{B \in \mathrm{Ob}(\mathcal{C})}$$

that is functorial in B, i.e. $\Phi_{B'} \circ f_* = f_* \circ \Phi_B$ for every $f : B \to B'$, then A and A' are isomorphic.

Formulate and prove analogous results for the functor $h_A = \mathrm{Hom}_{\mathcal{C}}(-, A)$.

Remark: These statements are often referred to as the *weak Yoneda lemma.*

Exercise 4.
Show that right adjoint functors preserve products and left adjoints functors preserve coproducts.

This means that if $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ is right adjoint to $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ and $\prod X_i$ together with morphisms $\mathrm{pr}_j : \prod X_i \to X_j$ (for $j \in I$) is a product of a family $\{X_i\}_{i \in I}$ of objects in $\mathcal{C}$, then $\mathcal{G}(\prod X_i)$ together with the morphisms $\mathcal{G}(\mathrm{pr}_j) : \mathcal{G}(\prod X_i) \to \mathcal{G}(X_j)$ (for $j \in I$) is a product of the family $\{\mathcal{G}(X_i)\}_{i \in I}$ in $\mathcal{D}$. And similar for the coproduct of a family $\{Y_i\}_{i \in I}$ of objects in $\mathcal{D}$.

Hint: Use Exercise 3.

Exercise 5.
Let A and B be rings and $\mathcal{G} : A - \mathrm{Mod} \to B - \mathrm{Mod}$ and $\mathcal{F} : B - \mathrm{Mod} \to A - \mathrm{Mod}$ adjoint functors with natural isomorphism $\Phi : \mathrm{Hom}_A(\mathcal{F}(-), -) \to \mathrm{Hom}_B(-, \mathcal{G}(-))$.

1. Show that $\mathcal{F}$ and $\mathcal{G}$ are additive functors and that $\Phi_{N,M} : \mathrm{Hom}_A(\mathcal{F}(N), M) \to \mathrm{Hom}_B(N, \mathcal{G}(M))$ is a group isomorphism for all A-modules M and B-modules N.
2. Show that $\mathcal{F}$ is right exact. In particular, $\mathcal{F}$ preserves cokernels.
3. Show that $\mathcal{G}$ is left exact. In particular, $\mathcal{G}$ preserves kernels.
4. Show that the functor
$$\mathrm{Hom}_A(M, -) : A - \mathrm{Mod} \longrightarrow A - \mathrm{Mod}$$
is right adjoint (to which functor?).
5. Show that the functors
$$- \otimes_A M : A - \mathrm{Mod} \longrightarrow A - \mathrm{Mod} \quad \text{and} \quad - \otimes_A B : A - \mathrm{Mod} \longrightarrow B - \mathrm{Mod}$$
are left adjoint (to which functors?).