
Exercise 1.
Describe all ring homomorphisms $\mathbb{R}[T] \to \mathbb{R}[T]$. Which of them are isomorphisms?

Exercise 2.

1. Let A be a ring and $\{I_i\}_{i\in I}$ be a family of ideals in A. Show that the intersection $\bigcap_{i\in I} I_i$ is an ideal of A. For finite I, show that the product $\prod_{i\in I} I_i = \big(\{\prod a_i | a_i \in I_i\}\big)$ is an ideal of A that is contained in $\bigcap_{i\in I} I_i$. Under which assumption is $\prod I_i = \bigcap I_i$?
2. Let $f : A \to B$ be a ring homomorphism and I an ideal of B. Show that $f^{-1}(I)$ is an ideal of A. Show that $f^{-1}(I)$ is prime if I is prime. Is $f^{-1}(I)$ maximal if I is maximal? Is the image $f(J)$ of an ideal J of A an ideal of B?

Exercise 3.

1. Describe all ideals of $\mathbb{Z}$. Which of them are principal ideals, which of them are prime and which of them are maximal?
2. Let $f \in \mathbb{R}[T]$ be of degree ≤ 2. When is (f) a prime ideal, when is it a maximal ideal? When is the quotient ring isomorphic to $\mathbb{R}$? When is it isomorphic to $\mathbb{C}$?

Exercise 4.
Let A be a ring and $A^\times$ its unit group.

1. Show that the map
$$\begin{array}{ccc} A^\times \times A^\times & \longrightarrow & A^\times \\ (a,b) & \longmapsto & ab \end{array}$$
is well-defined and turns $A^\times$ into an abelian group.
2. Let $f : A \to B$ be a ring homomorphism. Show that $f(A^\times) \subset B^\times$ and that the restriction $f|_{A^\times} : (A^\times, \cdot\,) \to (B^\times, \cdot\,)$ of f is a group homomorphism.
3. Show that the map $A^\times \times A \to A$, defined by $(a,b) \mapsto ab$ is a group action of $(A^\times, \cdot\,)$ on A.
4. Show that $(a) = A$ if and only if $a \in A^\times$.
5. Let A be an integral domain. Consider the map $\Phi : A \to \{\text{ideals of } A\}$ that sends a to the principal ideal (a) of A. Show that $\Phi(a) = \Phi(b)$ if and only if a and b are contained in the same orbit of the action of $A^\times$ on A.

Exercise 5 (Bonus exercise)**.**
Show that the ring $\mathbb{Z}$ satisfies the following universal property: for every ring A, there is a unique ring homomorphism $f : \mathbb{Z} \to A$. Use this and the universal property of the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to show that for a ring A whose underlying additive group $(A, +)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, +)$ (as a group), there is a unique ring isomorphism $\mathbb{Z}/n\mathbb{Z} \to A$.
Remark: We say that $\mathbb{Z}$ is an *initial object in the category of rings.*

Exercise 6 (Bonus exercise)**.**
Let A be a ring and I an ideal of A. Show that I is contained in a maximal ideal of A.
Hint: This is a consequence of Zorn's lemma.