
Exercise 1.
Let $e_1, \ldots, e_n$ be pairwise coprime positive integers. Show that the underlying additive group of $\mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_n\mathbb{Z}$ is a cyclic group.

Exercise 2.
Let A be an integral domain and $a, b, c, d, e \in A$.

1. Show that if d is a greatest common divisor of b and c and e is a greatest common divisor of ab and ac, then $(e) = (ad)$. Conclude that $\gcd(ab, ac) = (a) \cdot \gcd(b, c)$.
2. If A is a principal ideal domain, then d is a greatest common divisor of a and b if and only if $(a, b) = (d)$. Conclude that every two elements of a principal ideal domain have a greatest common divisor.
3. Find an integral domain A with elements $a, b, d \in A$ such that d is a greatest common divisor of a and b, but $(a, b) \neq (d)$.

Exercise 3.
Let $\mathbb{Z}[\sqrt{-5}]$ be the set of complex numbers of the form $z = a + b\sqrt{-5}$ with $a, b \in \mathbb{Z}$ and $\sqrt{-5} = i\sqrt{5}$.

1. Show that $\mathbb{Z}[\sqrt{-5}]$ is a subring of $\mathbb{C}$.
2. Show that the association $a + b\sqrt{-5} \mapsto a^2 + 5b^2$ defines a map $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ with $N(zz') = N(z)N(z')$ and $N(1) = 1$.
   Remark: $N(z)$ is the square of the usual absolute value of the comlex number z.
3. Conclude that $z \in \mathbb{Z}[\sqrt{-5}]^\times$ if and only if $N(z) \in \mathbb{Z}^\times$. Determine $\mathbb{Z}[\sqrt{-5}]^\times$.
4. Show that 2, 3, $(1 + \sqrt{-5})$ and $(1 - \sqrt{-5})$ are irreducible, but not prime.
5. Show that 6 and $2 + 2\sqrt{-5}$ do not have a greatest common divisor.

Exercise 4.

1. Determine all units, prime elements and irreducible elements of $\mathbb{Z}/6\mathbb{Z}$.
2. Let $\mathbb{R}[T_1, T_2] = \big(\mathbb{R}[T_1]\big)[T_2]$ be the polynomial ring over $\mathbb{R}$ in T_1 and T_2 and I the ideal generated by $T_1^2 + T_2^2$. Is the class $\overline{T}_1 = T_1 + I$ a prime element in the quotient ring $\mathbb{R}[T_1, T_2]/I$? Is $\overline{T}_1$ irreducible?

Exercise 5 (Bonus exercise)**.**
Prove the *fundamental theorem of algebra*: given a polynomial $f \in \mathbb{C}[T]$ of positive degree, then there exists a $z \in \mathbb{C}$ such that $f(z) = 0$.