
Exercise 1.

1. Let $f \in \mathbb{R}[T]$ be a polynomial without a root in $\mathbb{R}$. Show that f has even degree.
2. Show that $T^5 + 4T + 14$ is without a root in $\mathbb{Q}$, though it has odd degree.

Exercise 2.
Let A be a unique factorization domain and $K = \text{Frac}A$. Let $f = \sum_{i=0}^{n} a_i T^i$ be a polynomial of degree n in $A[T]$ and $a \in K$ a root of f. Write $a = \frac{b}{c}$ such that 1 is a greatest common divisor of $b, c \in A$. Show that b is a divisor of a_0 and c is a divisor of a_n. In particular, if $a_n = 1$, then $a \in A$ and a is a divisor of a_0.

Exercise 3.
Show that the following polynomials are irreducible in $\mathbb{Q}[T]$.

- $T^2 + 1$;
- $2T^4 - 18T - 12$;
- $T^3 + T^2 + 1$;
- $T^3 - 13T + 5$;
- $T^5 + 3T^3 + 6T^2 + 1$;
- $T^{12} - 2$.

Which of them are irreducible in $\mathbb{Z}[T]$? Which of them are irreducible in $\mathbb{Z}[i][T]$?

Exercise 4.
For which $a \geq 0$ is $T^4 + aT^2 + 1$ irreducible in $\mathbb{Z}[T]$?

Exercise 5.
Let p be a prime number. Show that $T^{p-1} + \cdots + T + 1$ is irreducible in $\mathbb{Z}[T]$.
Hint: Show that $f(T+1)$ is irreducible and conclude that $f = f(T)$ is irreducible.

Exercise 6.

1. Find a greatest common divisor of $T^4 + T^3 + 2T^2 + T + 1$ and $T^3 + 4T^2 + 4T + 3$ in $\mathbb{Q}[T]$.

2. Find a greatest common divisor of $4T^5 + 7T^3 + 2T^2 + 1$ and $3T^3 + T + 1$ in $\mathbb{Q}[T]$.

Exercise 7.
Let $a + bi \in \mathbb{Z}[i]$.

1. Show that $(a + bi)^{-1} = \frac{a-bi}{a^2+b^2}$, as an element of $\mathbb{Q}[i] = \operatorname{Frac} \mathbb{Z}[i]$.

2. Find a greatest common divisor of $3 + 2i$ and $2 - i$.

3. Find a greatest common divisor of $8 + 9i$ and $-1 + 7i$ in $\mathbb{Z}[i]$.

Exercise 8.
Which of the following rings are integral domains, unique factorization domains, principal ideal domains, Euclidean rings, fields or local rings?

1. $\mathbb{Z}[i]$;

2. $\mathbb{Z}[\sqrt{-5}]$;

3. $\mathbb{Z}_{(2)} = \{\frac{a}{b} \in \mathbb{Q} | a, b \in \mathbb{Z}$ and b odd$\}$;

4. $\mathbb{R}[T]/(T^2 + 1)$;

5. $\mathbb{C}[T]/(T^2 + 1)$;

6. $S^{-1}\mathbb{C}[T]$ for $S = \{\sum a_i T^i \in \mathbb{C}[T] | a_0 \neq 0\}$;

7. $\mathbb{C}[T_1, T_2, T_3, T_4]/(T_1T_2 - T_3T_4)$.

Exercise 9.
Show that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$. Show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean ring and that it is isomorphic to $\mathbb{Z}[T]/(T^2 - 2)$.

Exercise 10.
Give three examples of

1. an integral domain that is not a unique factorization domain;

2. a unique factorization domain that is not a principal ideal ring;

3. a local principal domain that is not a field;

4. a local ring that is not an integral domain.

Exercise 11.
Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with two elements 0 and 1.

1. Show that $f = T^2 + T + 1$ is an irreducible polynomial in $\mathbb{F}_2[T]$.
2. Show that $\mathbb{F}_4 = \mathbb{F}_2[T]/(f)$ is a field with four elements.
3. Show that $\mathbb{F}_4^\times$ is a cyclic group with 3 elements.
4. Show that $T^4 - T = \prod_{a \in \mathbb{F}_4}(T - a)$ (as a polynomial in $\mathbb{F}_4[T]$).
5. Find a factorization of $T^4 - T$ in $\mathbb{F}_2[T]$.

Remark: The polynomial f is called the *third cyclotomic polynomial*.

Exercise 12.
Let A be a ring.

1. If ideals I and J are coprime then, also I^m and J^n are coprime for all $m, n \geq 1$.
2. Given pairwise coprime ideals I_i and elements a_i for $i = 1, \dots, n$, there exits an $x \in A$ such that $x - a_i \in I_i$.

Exercise 13.
Let A be a ring and let $n\mathbb{Z}$ be the kernel of the unique ring homomorphism $\mathbb{Z} \to A$ where $n \geq 0$. The number $\mathrm{char} A = n$ is called the *characteristic of* A.

1. Show that if n is positive, then n is the smallest positive integer such that
$$n \cdot 1 \ = \ \underbrace{1 + \cdots + 1}_{n-\text{times}} \ = \ 0.$$
If $n = 0$, then $k \cdot 1 \neq 0$ for any $k \geq 0$.
2. Show that n is zero or a prime number if A is an integral domain.
3. Let L/K be a field extension. Show that K and L have the same characteristic.
4. Let K be a field of characteristic 0. Show that there is a unique ring homomorphism $\mathbb{Q} \to K$.
5. Let p be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. Let K be a field of characteristic p. Show that there is a unique ring homomorphism $\mathbb{F}_p \to K$.
6. Give an example of a ring homomorphism $A \to B$ where A and B have different characteristics.

Remark: The image of the unique homomorphism $\mathbb{Q} \to K$ (if $\mathrm{char} K = 0$) or $\mathbb{F}_p \to K$ (if $\mathrm{char} K = p > 0$) is called the *prime field of* K.

Exercise 14.
Let $f : A \to B$ be a surjective ring homomorphism and A a local ring. Show that B is also a local ring.

Exercise 15.
Let A be a ring. Show that the set of all zero divisors of A is a union of ideals.
Hint: The kernel I_a of the multiplications map $m_a : A \to A$ by a consists of zero divisors if $a \neq 0$.

Exercise 16.
Prove the following stronger version of the statement of the previous exercise: the set of all zero divisors of a ring A is a union of prime ideals. This can be done along the following lines.

1. Let I_a be the kernel of the multiplication map $m_a : A \to A$. Show that $I_0 = A$ and that all elements of I_a are zero divisors if $a \neq 0$.
2. Let $\mathcal{S} = \{I_a | a \in A - \{0\}\}$, which is partially ordered by inclusion. Show that for every $I_a \in \mathcal{S}$, there is a maximal element I_b in $\mathcal{S}$ containing I_a.
3. Show that the set of zero divisors is the union of all maximal elements I_a of $\mathcal{S}$.
4. Show that $I_a \subset I_{ab}$ for any $a, b \in A$.
5. Show that every maximal element I_a in $\mathcal{S}$ is a prime ideal.