Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 7 Oliver Lorscheid
To hand in at 4.5. in the exercise class Esteban Arreaga (monitor)

Exercise 1.
Let K be a field.

1. Show that in the categories Ab, $K-$Vect and Ring, a morphism is an isomorphism if and only if it is a bijective map.
2. Show that in the categories Ab, $K-$Vect and Ring, a morphism is a monomorphism if and only if it is an injective map.
3. Show that in the categories Ab and $K-$ Vect, a morphism is an epimorphism if and only if it is a surjective map.
4. Show that there are ring homomorphisms $\alpha : A \to B$ that are monomorphisms and epimorphisms, but not isomorphisms in the category Ring.

Exercise 2.

1. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor and $\alpha : A \to B$ an isomorphism in $\mathcal{C}$. Show that $\mathcal{F}(\alpha)$ is an isomorphism in $\mathcal{D}$.
2. Give an example of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and an epimorphism α in $\mathcal{C}$ such that $\mathcal{F}(\alpha)$ is not an epimorphism.
3. Give an example of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and a monomorphism α in $\mathcal{C}$ such that $\mathcal{F}(\alpha)$ is not a monomorphism.

Exercise 3.
Let $\mathcal{C}$ be a category and $\{A_i\}_{i\in I}$ a family of objects in $\mathcal{C}$. Assume that $\mathcal{C}$ has a product $\prod_{i\in I} A_i$ and a coproduct $\coprod_{i\in I} A_i$. Let B be another object of $\mathcal{C}$.

1. Show that there is a bijection $\mathrm{Hom}_{\mathcal{C}}\Big(B, \prod\limits_{i\in I} A_i\Big) \longrightarrow \prod\limits_{i\in I} \mathrm{Hom}_{\mathcal{C}}(B, A_i)$.
2. Show that there is a bijection $\mathrm{Hom}_{\mathcal{C}}\Big(\coprod\limits_{i\in I} A_i, B\Big) \longrightarrow \prod\limits_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A_i, B)$.

Exercise 4.
Let $\mathcal{C}$ be a category and $\alpha : A \to B$ a morphism in $\mathcal{C}$. A *(categorical) image of* α is an object $\mathrm{im}(\alpha)$ in $\mathcal{C}$ together with a morphism $\pi : A \to \mathrm{im}(\alpha)$ and a monomorphism $\iota : \mathrm{im}(\alpha) \to B$ such that $\alpha = \iota \circ \pi$ that satisfy the following universal property: for every object C, morphism $\pi' : A \to C$ and monomorphism $\iota' : C \to B$ such that $\alpha = \iota' \circ \pi'$ there is a unique morphism $\beta : \mathrm{im}(\alpha) \to C$ such that $\pi' = \beta \circ \pi$ and $\iota = \iota' \circ \beta$.

1. Draw a diagram taking all the above objects and morphisms into consideration.
2. Assume that the image $\mathrm{im}(\alpha)$ of α exits. Show that if C together with $\pi' : A \to C$ and $\iota' : C \to B$ is an image of α, then there is a unique isomorphism $\beta : \mathrm{im}(\alpha) \to C$ such that$\pi' = \beta \circ \pi$ and $\iota = \iota' \circ \beta$.
3. Let $\alpha : A \to B$ be a morphism in Set. Consider the set
$$\mathrm{im}(\alpha) \quad = \quad \{\, b \in B \,|\, b = \alpha(a) \text{ for some } a \in A \,\},$$
and the maps $\pi : A \to \mathrm{im}(\alpha)$ with $\pi(a) = \alpha(a)$ and $\iota : \mathrm{im}(\alpha) \to B$ with $\iota(b) = b$. Show that $\mathrm{im}(\alpha)$ together with π and ι is a categorical image of α in Set.
4. Show that the analogous statements to part 3 hold for Ab, $K -$ Vect and Ring.

Exercise 5 (Bonus)**.**

1. Look up the definition of a topological space and of a continuous map.
2. Show that Top has initial and terminal objects as well as products and coproducts.

Recall from Exercise 5 from List 4 the definition of Spec$A$ as the set of all prime ideals $\mathfrak{p}$ of A together with the topology generated by the principal open subsets U_a where a varies through all elements of A. For a ring homomorphism $f : A \to B$, we define $f^* = \mathrm{Spec} f$ as the map that sends a prime ideal $\mathfrak{p}$ of Spec$B$ to $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$.

3. Show that this defines a contravariant functor Spec : Ring $\to$ Top.
4. Show that the spectrum of the product of a finite number of rings is isomorphic to the coproduct of their spectra.