Exercises for Algebra 1 — Instituto Nacional de Matemática Pura e Aplicada
List 9 — Oliver Lorscheid
To hand in at 25.5. in the exercise class — Esteban Arreaga (monitor)

Exercise 1. Let M, N, N_i and P be A-modules where $i \in I$ for some index set I. Verify the following properties of the tensor product:

(1) $M \otimes_A A \simeq M$;
(2) $M \otimes_A N \simeq N \otimes_A M$;
(3) $(M \otimes_A N) \otimes_A P \simeq M \otimes_A (N \otimes_A P)$;
(4) $M \otimes_A \left(\bigoplus_{i\in I} N_i\right) \simeq \bigoplus_{i\in I} (M \otimes_A N_i)$.

Exercise 2. Let k be a field, $A = k[T]$ and $M = N = k^2$ the A-modules from Exercise 2 of List 8. Let $P = k$.

1. Show that the map $A \times P \to P$ with $\left(\sum a_i T^i\right).(m) = \sum a_i.m$ defines an A-module structure for P.
2. Show that the inclusion $a \mapsto (a, 0)$ into the first coordinate defines injective A-linear maps $i : P \to M$ and $j : P \to N$.
3. Show that there are short exact sequences of the form

   $$0 \longrightarrow P \xrightarrow{i} M \xrightarrow{p} P \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow P \xrightarrow{j} N \xrightarrow{q} P \longrightarrow 0$$

   for some A-linear maps p and q.
4. Which of these sequences are split?

Exercise 3. Let $0 \to V_1 \to \cdots \to V_n \to 0$ be an exact sequence of k-vector spaces. Show that $\sum (-1)^k \dim_k V_k = 0$.

Exercise 4.
Let $f : M \to N$ be an A-linear homomorphism and consider its factorization $f = i \circ p$ into the surjection $p : M \to \mathrm{im} f$ and the inclusion $i : \mathrm{im} f \to N$.

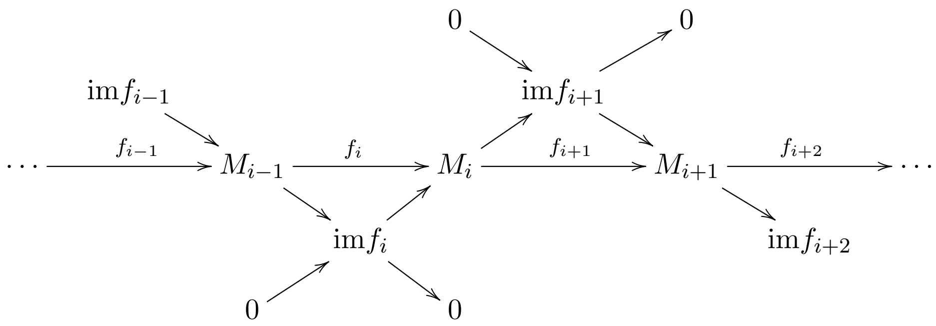
1. Show that $i : \mathrm{im} f \to N$ is the kernel of the cokernel $N \to \mathrm{coker} f$ of f
2. Show that $p : M \to \mathrm{im} f$ is the cokernel of the kernel $\ker f \to M$ of f.
3. Show that $\mathrm{im} f$ together with $p : M \to \mathrm{im} f$ and $i : \mathrm{im} f \to N$ is a categorical image of f (see Exercise 4 on List 7).
4. Given an exact sequence

   $$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \cdots$$

   with $i \in I$, show that we obtain for every $i \in I$ a short exact sequence

   $$0 \longrightarrow \mathrm{im} f_i \longrightarrow M_i \longrightarrow \mathrm{im} f_{i+1} \longrightarrow 0,$$

   and that these sequences fit together to a commutative diagram



Exercise 5 (Bonus). Prove the *short 5-lemma*: given a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_1 & \xrightarrow{i_1} & M_1 & \xrightarrow{p_1} & Q_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_N} & & \downarrow{\scriptstyle f_M} & & \downarrow{\scriptstyle f_Q} & & \\
0 & \longrightarrow & N_2 & \xrightarrow[i_2]{} & M_2 & \xrightarrow[p_2]{} & Q_2 & \longrightarrow & 0
\end{array}$$

with exact rows, then

1. f_M is a monomorphism if f_N and f_Q are monomorphisms,
2. f_M is an epimorphism if f_N and f_Q are epimorphisms, and
3. f_M is an isomorphism if f_N and f_Q are isomorphisms.