Exercises for Algebraic Number Theory
List 6 to hand in at 13.2.2017 in the exercise class

Exercise 1.
Show that every Dedekind domain with a finite number of maximal ideals is a principal domain.

Exercise 2.
Find an extensions L/K of number fields with Galois group G and respective rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$ for each of the following requirements.

1. The decomposition group $G_\mathfrak{q}$ of some prime ideal $\mathfrak{q}$ of $\mathcal{O}_L$ over $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}_K$ is not a normal subgroup of G.
2. $G \simeq I_\mathfrak{q} \times I_{\mathfrak{q}'}$ is the direct product of two nontrivial inertia subroups $I_\mathfrak{q}$ and $I_{\mathfrak{q}'}$ where $\mathfrak{q}$ and $\mathfrak{q}'$ are prime ideals of $\mathcal{O}_L$.
3. The inertia subgroup $I_\mathfrak{q}$ is not cyclic for a prime ideal $\mathfrak{q}$ of $\mathcal{O}_L$.

Using the same notation as above, show that the quotient $G_\mathfrak{q}/I_\mathfrak{q}$ is cyclic for all extensions of number fields.

Exercise 3.
Let p be a prime number such that $p-1$ is divisible by 3 and such that 2 is a cube modulo p. Let ζ_p a p-th root of unity and $G = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ the Galois group of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$.

1. Show that such a prime number p exists.
2. Show that there is a unique subfield K of $\mathbb{Q}(\zeta_p)$ that has degree 3 over $\mathbb{Q}$.
3. Show that 2 is unramified in $\mathbb{Q}(\zeta_p)$ and that the inertia degree divides $(p-1)/3$. Conclude that 2 splits completely in K.
4. Conclude that if $\mathcal{O}_K \neq \mathbb{Z}[a]$ for any element $a \in \mathcal{O}_K$.
   Hint: The minimal polynomial of a cannot have three different zeros modulo 2.

Exercise 4.
Determine the decomposition of the primes 2, 3, 5 and 7 in $\mathbb{Q}(\zeta_{12})$. Find the decomposition fields and inertia fields for all prime ideals in $\mathbb{Z}[\zeta_{12}]$ in these decompositions. Show that $\mathbb{Q}(\zeta_{12})$ has trivial class group.

Advanced bonus exercise: Show that $\mathbb{Q}(\zeta_{15})$ has trivial class group.

***Exercise 5.**
Let A be a Dedekind domain, $K = \text{Frac}A$, L/K a finite field extension and B the integral closure of A in L. Let I be an ideal of A and J be an ideal of B. Show that IB is coprime to J (as ideals of B) if and only if I is coprime to $J \cap A$ (as ideals of A). What does this mean for the hypothesis of Proposition 1 of section 1.7 of the lecture?

***Exercise 6.**
Show that

1. the p-adic absolute value $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ where p is a prime number,
2. the f-adic absolute value $|\cdot|_f : K_0(T) \to \mathbb{R}_{\geq 0}$ where $f \in K_0[T]$ is irreducible, and
3. the "infinite absolute value" $|\cdot|_\infty : K_0(T) \to \mathbb{R}_{\geq 0}$

are indeed absolute values. Show further that $\{1/n\}_{n\geq 1}$ does not converge w.r.t. any p-adic absolute value. What is the analogue statement for $K_0(T)$?

***Exercise 7.**
Let K be a field with absolute value $|\cdot|$. Show that $|\cdot|$ is non-archimedean if and only if the set of all $x \in K$ such that $|x| \leq 1$ forms an additive subgroup. Show that the same is true if we require a strict inequality $|x| < 1$.

***Exercise 8.**
Let K be a complete field with discrete valuation v, valuation ring $\mathcal{O}_v$ and residue field k_v. Show that the canonical projection $\mathcal{O}_v \to k_v$ induces a multiplicative bijection $\mu(K) \to \mu(k_v)$.

***Exercise 9.**
Let K be a field with absolute value $|\cdot|$, and $x \in K$.

1. Show that $|x| = 1$ if $x^n = 1$ for some $n \geq 1$.
2. Show that $|-x| = |x|$.

***Exercise 10.**
Let K be a field with two absolute values $|\cdot|_1$ and $|\cdot|_2$. Show that the following properties are equivalent.

1. $|\cdot|_1$ and $|\cdot|_2$ define the same topology on K.
2. $|x|_1 < |y|_1$ if and only if $|x|_2 < |y|_2$ for all $x, y \in K$.
3. There exists an $s \in \mathbb{R}_{\geq 0}$ such that $|x|_1 = |x|_2^s$ for all $x \in K$.

***Exercise 11.**
Prove Theorem 2 from section 2.1 and Theorem 1 from section 2.3 of the lecture.

The starred exercises are not to hand in.