Exercises for Algebraic Number Theory
List 7 (not to hand in)

Exercise 1.
Show that

$$\begin{array}{ccc} \mathbb{Z}[[T]] & \longrightarrow & \mathbb{Z}_p \\ \sum_{i\geq 0} a_i T^i & \longmapsto & \sum_{i\geq 0} a_i p^i \end{array}$$

with $a_i \in \{0, \dots, p-1\}$ is a well-defined homomorphism of rings that is surjective and has kernel $(T-p)$.

Exercise 2.
Show that the p-adic integer $\sum_{i\geq 0} a_i p^i$ is in $\mathbb{Z}_p^\times$ if and only if $a_0 \neq 0$.

Hint: Use Hensel's Lemma.

Exercise 3.
$\mathbb{Q}_p$ and $\mathbb{Q}_q$ are isomorphic (as fields) for two prime numbers p and q if and only if $p = q$.
Hint: Make use of the different splitting behaviours of polynomials in $\mathbb{Z}[T]$ modulo p and q, respectively.

Exercise 4.
Determine all places of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{5})$.

Exercise 5.
Determine all archimedean places of $\mathbb{Q}(\zeta_n)$ and $\mathbb{Q}(\sqrt[n]{2})$ where $n \geq 1$ and ζ_n is a primitive n-th root of unity.

Exercise 6.
Let L/K be a finite separable extension, v a discrete valuation of K with valuation ring $A = \mathcal{O}_v$ and B the integral closure of A in L. Let $\hat{v}$ be the extension of v to the completion K_v of K at v, and $\mathcal{O}_{\hat{v}}$ its valuation ring. Show that

$$B \otimes_A \mathcal{O}_{\hat{v}} \;=\; \prod_{w|v} \mathcal{O}_{\hat{w}}$$

where $w|v$ varies over all extensions w of v to L and $\mathcal{O}_{\hat{w}}$ is the valuation ring of the extension $\hat{w}$ of w to the completion L_w of L at w.

Exercise 7.
What is the relation of Proposition 1 in section 2.6 to Proposition 1 in section 1.7 of the lecture?

*Exercise 8.
Let K be a field of positive characteristic p.

1. Show that $K_0((T))/K_0(T)$ is not an algebraic extension.
2. Let $X \in K_0((T))$ be transcendental over $K_0(T)$, $K = K_0(T, X^p)$ and $L = K_0(T, X)$. Show that L/K has degree p and that the fundamental identity does not hold for the T-adic valuation v_T of K.

Hint: Use that v_T extends uniquely to $K_0((T))$.

*Exercise 9.
Let A be a Dedekind domain, K its fraction field, L/K a finite separable field extension of degree n and B the integral closure of A in L. Let $\mathfrak{q} \subset B$ be a nonzero prime ideal such that $k(\mathfrak{q}) = B/\mathfrak{q}$ is separable over $k(\mathfrak{p}) = A/\mathfrak{p}$ where $\mathfrak{p} = \mathfrak{q} \cap A$. Show that there is an element $b \in B$ such that $L = K(b)$ and the conductor $\mathfrak{f}$ of $A[b]$ is coprime to $\mathfrak{q}$.

Hint: This is exercise 2 in Chapter III of Serre's book Local Fields.

*Exercise 10.
Let K be a field with a non-archimedean absolute value $|\cdot|$. For a polynomial $f = a_nT^n + \cdots + a_0$ in $K[T]$, define

$$|f| \;=\; \max\{|a_0|, \ldots, |a_n|\}.$$

Show that with this definition, $|\cdot|$ extends uniquely to a non-archimedean absolute value of $K(T)$.
Hint: $|fg| = |f| \cdot |g|$ can be proven in analogy to Gauss' lemma.

*Exercise 11.
Prove all basic facts from section 2.4.

*Exercise 12.
Determine for some (or all) $n \leq 10$ the prime decomposition of 2, 3, 5 and ∞ in $\mathbb{Q}(\zeta_n)$ where ζ_n is a primitive 12-th root of unity. In particular, determine the different places above 2, 3, 5 and ∞, their ramification indices and their inertia degrees. Calculate the decompostion and inertia fields of the primes above 2, 3 and 5. Show that the class number of $\mathbb{Q}(\zeta_n)$ is 1.

*Exercise 13.
Calculate the class group of $\mathbb{Q}(\sqrt{7})$.
Hint: $2 = (3+\sqrt{7})(3-\sqrt{7})$ and $-1+\sqrt{7} = (2+\sqrt{7})(3-\sqrt{7})$.

*Exercise 14.
Let L/K be a Galois extension of number field with non-cyclic Galois group. Show that all places of K but a finite number of exceptions have more than one extension to L.
Hint: Recall that the Galois groups of extensions of finite fields are cyclic.

*__Exercise 15.__ Let p be a prime number.

1. Let $a \in \mathbb{Q}_p^\times$. Prove that $a \in \mathbb{Z}_p^\times$ if and only if a has a n-th root in $\mathbb{Q}_p$ for infinitely many $n \geq 0$, i.e. $a = b^n$ for some $b \in \mathbb{Q}_p$.

2. Prove that the only field automorphism of $\mathbb{Q}_p$ is the identity map.

Hint: Use 1 to show that every field automorphism must map $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p^\times$.