Exercises for Commutative Algebra
List 10 to hand in at 29.5.2017

Exercise 1.
Show that every Noetherian unique factorization domain of dimension 1 is a principal ideal domain. Conclude that a Dedekind domain is a unique factorization domain if and only if it is a principal ideal domain.

Exercise 2.
Let A be a Dedekind domain and $I \subset A$ a nonzero ideal. Show that every ideal in A/I is principal. Conclude that every ideal in A can be generated by at most 2 elements.

Exercise 3.
Let k be an algebraically closed field. Show that $\dim k[X,Y] = 2$ along the following steps.

1. Show that $\dim k[X,Y] \geq 2$.
2. Consider a maximal chain $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_l$ of prime ideals in $k[X,Y]$. Reason that $\mathfrak{p}_0 = (0)$ and $\mathfrak{p}_l = (X-a, Y-b)$ for some $a, b \in k$.
3. Let $\mathfrak{q}_i = \mathfrak{p}_i \cap k[X]$. Show that there is some k such that $\mathfrak{q}_i = (0)$ for $i = 0, \ldots, k$ and $\mathfrak{q}_i = (X-a)$ for $i = k+1, \ldots, l$.
4. Show that k is 0 or 1 by localizing $k[X,Y]$ and $k[X]$ at the common multiplicative subset $S = k[X] - \{0\}$ and observing that $S^{-1}k[X,Y]$ is a principal ideal domain.
5. Conclude that $l = 2$ by dividing $k[X,Y]$ by $(X-a)$, which yields, once again, a principal ideal domain.

Note that this proof shows the stronger fact that every maximal chain of prime ideals in $k[X,Y]$ has length 2.

Exercise 4.
Let k be an algebraically closed field and $f \in k[X,Y]$ irreducible. Show that $A = k[X,Y]/(f)$ is a Dedekind domain if and only if $(f, \partial f/\partial X, \partial f/\partial Y) = k[X,Y]$.

Exercise 5.
Let k be an algebraically closed field, $f = Y^2 - X^3 - X^2$ and $A = k[X,Y]/(f)$. Verify that A is a Noetherian domain of dimension 1. Find all singularities of the plane affine curve $C = \overline{Z}(Y^2 - X^3 - X^2)$. Show that the homomorphism

$$\begin{array}{rccc} f: & A & \longrightarrow & k[T] \\ & [X] & \longmapsto & T^2-1 \\ & [Y] & \longmapsto & T(T^2-1) \end{array}$$

is well-defined and defines an integral extension of rings that induces an isomorphism $\tilde{f} : \operatorname{Frac} A \to \operatorname{Frac} k[T] = k(T)$ between the respective fraction fields. Conclude that $\tilde{f}$ identifies the integral closure of A in $\operatorname{Frac} A$ with $k[T]$. Show that there is a nonsingular plane affine curve $\overline{C}$ with coordinate ring isomorphic to $k[T]$. Describe a morphism of affine varieties $\varphi : \overline{C} \to C$ whose associated homomorphism of coordinate rings is f. Show that φ is the normalization of C. Make an illustration of φ.