Exercises for Commutative Algebra
List 11 to hand in at 5.6.2017

Exercise 1.
Find an example of a Noetherian ring that has infinite dimension, and prove this.

Hint: Atiyah and Macdonald's book contains such an example (exercise 4 in chapter 11).

Exercise 2.
Let A be a ring. Show that $\dim A + 1 \leq \dim A[T] \leq 2 \dim A + 1$.

Hint: See exercise 6 of chapter 11 in Atiyah and Macdonald's book for hints.

Exercise 3.
Let A be a ring, I an ideal of A and M an A-module. Let $\alpha : A \to \widehat{A}$ and $\beta : M \to \widehat{M}$ be the canonical maps to the I-adic completions of A and M, respectively.

1. Show that M is Hausdorff in the I-adic topology if and only if $\bigcap_{n \geq 0} I^n.M = \{0\}$, which is the case if and only if β is injective.
2. Define $\widehat{I} = \{ (a_i) \in \widehat{A} \,|\, a_1 \equiv 0 \mod I \}$. Show that $\widehat{I}$ equals the topological closure of $\alpha(I)$ in $\widehat{A}$ and that $\widehat{I}$ is an ideal of $\widehat{A}$.
3. Show that β induces isomorphisms $M/I^n.M \to \widehat{M}/\big(\widehat{I}\big)^n.\widehat{M}$ of A-modules.
4. Show that the $\widehat{I}$-adic topology of $\widehat{M}$ is Hausdorff and that the $\widehat{I}$-adic completion $\hat{\beta} : \widehat{M} \to \widehat{\widehat{M}}$ is an isomorphism of $\widehat{A}$-modules.

Exercise 4.
Let A be a ring and I an ideal of A such that A is Hausdorff with respect to the I-adic topology. Let q be a real number in $(0, 1)$.

1. Show that the function $d(a, b) = \inf\{q^n | a - b \in I^n\}$ is a metric on A.
2. Let $\widehat{A}^d$ be the completion of A as a metric space w.r.t. d. Exhibit a bijection $f : \widehat{A} \to \widehat{A}^d$ that is a homeomorphism where we consider $\widehat{A}$ with the $\widehat{I}$-adic topology and that is an isomorphism of rings where we consider $\widehat{A}^d$ with the componentwise addition and multiplication of Cauchy sequences.

Exercise 5.
Let A be a Noetherian ring, I an ideal of A and $S = 1 + I$.

1. Show that S is a multiplicative subset of A and that the kernel of the localization $\iota : A \to S^{-1}A$ equals the kernel of the completion $\alpha : A \to \widehat{A}$.
2. Show that $\alpha(a)$ is invertible in $\widehat{A}$ for every $a \in S$.
3. Conclude that $\alpha : A \to \widehat{A}$ extends to an injective ring homomorphism $S^{-1}A \to \widehat{A}$.