Exercises for Commutative Algebra
List 12 to hand in at 12.6.2017

Exercise 1.
Let A be a Noetherian ring of finite dimension. Show that $\dim A[T] = \dim A + 1$. Conclude that $\dim k[T_1, \dots, T_n] = n$ if k is a field.

Exercise 2 (Geometric relevance of Krull's Hauptidealsatz)**.**
Let k be an algebraically closed field and A a finitely generated domain over k. Let $V \subset \mathbb{A}_k^n$ be an affine k-variety for some $n \geq 0$ with coordinate ring A. Let $f \in A$ be neither zero nor a unit. Show that the subvariety Z of V that is defined by f has dimension $\dim Z = \dim V - 1$. Conclude that $\dim Z \geq \dim V - r$ if Z is defined by r elements $f_1, \dots, f_r \in A$. Show that for $r \geq 2$, the condition $f_1, \dots, f_r \notin A^\times \cup \{0\}$ does not imply that $\dim Z = \dim V - r$.

Exercise 3 (Non-singular points of varieties)**.**
Let k be an algebraically closed field and $V \subset \mathbb{A}_k^n$ be an irreducible affine k-variety of dimension r with vanishing ideal $I = (f_1, \dots, f_r) \subset k[T_1, \dots, T_n]$ and coordinate ring $A = k[T_1, \dots, T_n]/I$. Let $x = \mathfrak{m}$ be a point of V, which is a maximal ideal of A. We say that V is *non-singular at x* if the Jacobian matrix

$$J \;=\; J_{f_1,\dots,f_r}(T_1, \dots, T_n) \;=\; \left(\frac{\partial f_i}{\partial T_j}\right)_{j=1,\dots,n,\, i=1,\dots,r}$$

has rank $n - r$ in $k(x) = A/\mathfrak{m}$. Show that V is non-singular in x if and only if $A_\mathfrak{m}$ is a regular local ring.

Hint: Generalize the corresponding proof for curves from the lecture.

Exercise 4.
Let k be an algebraically closed field. Let $V \subset \mathbb{A}_k^2$ be the variety defined by the polynomials $f_1 = X^2$ and $f_2 = XY$. Show that $\dim V = 1$ and that $x = \mathfrak{m} = (X, Y) \in V$. Show that V is non-singular in x and that the rank of $J_{f_1,f_2}(T_1, T_2)$ in $k(x)$ is 0. Why does this not contradict the claim of Exercise 3?

Exercise 5 (Equational criterium for flatness)**.**
Let A be a ring and M an A-module. Then A is flat if and only if for every relation of the form $\sum a_i.m = 0$ with $a_1, \dots, a_k \in A$ and $m_1, \dots, m_k \in M$, there are elements $n_1, \dots, n_l \in M$ and elements $b_{i,j} \in A$ for $i = 1, \dots, k$ and $j = 1, \dots, l$ such that

$$m_i \;=\; \sum_{j=1}^{l} b_{i,j}.n_j \quad \text{for all } i = 1, \dots, k \text{ and } \quad \sum_{i=1}^{k} a_i b_{i,j} \;=\; 0 \quad \text{for all } j = 1, \dots, l.$$