Exercises for Commutative Algebra
List 13 to hand in at 19.6.2017

Exercise 1.
Show that a $\mathbb{Z}$-module is injective if and only if it is divisible. Let $A \to B$ be a ring homomorphism and I an injective A-module. Show that $\operatorname{Hom}_A(B, I)$ is an injective B-module.

Exercise 2.
Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then the association

$$\begin{array}{rccc} L_i\mathcal{F} : & \mathcal{A} & \longrightarrow & \mathcal{B} \\ & M & \longmapsto & H_i\big(\mathcal{F}(P_\bullet)\big) \\ & [f : M \to N] & \longmapsto & \big[H_i(f_\bullet) : H_i\big(\mathcal{F}(P_\bullet)\big) \to H_i\big(\mathcal{F}(P'_\bullet)\big)\big] \end{array}$$

defines a functor for every i that does not depend on the choices of the projective resolutions $P_\bullet$ of M and $P'_\bullet$ of N and of the morphism $f_\bullet : P_\bullet \to P'_\bullet$ with $H_0(f_\bullet) = f$. Moreover, if $\mathcal{F}$ is right exact, then $\epsilon : P_0 \to M$ induces a natural equivalence $L_0\mathcal{F} \to \mathcal{F}$ of functors. If, in addition, M is projective, then $L_i\mathcal{F}(M) = 0$ for all $i > 0$.

Exercise 3.
Let $A = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$ and $M = A/(d)$ for some divisor d of n. Calculate $\operatorname{Tor}_i(M, N)$ for all A-modules N and all $i \geq 0$.

Exercise 4.
Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M be a finitely generated A-module. Show that the following are equivalent.

1. M is free.
2. M is flat.
3. The inclusion $\mathfrak{m} \to A$ induces an injective map $\mathfrak{m} \otimes_A M \to A \otimes_A M$.
4. $\operatorname{Tor}_1^A(k, M) = 0$.

Hint: Exercise 15 of chapter 7 in Atiyah-Macdonald's book contains hints.

Exercise 5.
Let A be a Noetherian ring and M a finitely generated A-module. Show that the following are equivalent.

1. M is projective.
2. M is flat.
3. $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of A.
4. There are elements $a_1, \dots, a_n \in A$ that generate the unit ideal $(1) = (a_1, \dots, a_n)$ such that $M[a_i^{-1}]$ is a free $A[a_i^{-1}]$-module for $i = 1, \dots, n$ where $M[a^{-1}] = S^{-1}M$ for $S = \{a^k\}_{k \geq 0}$.

Remark: We say that M is *locally free* if it satisfies property 4. This is closely connected to the notion of a vector bundle over $\operatorname{Spec} A$.