Exercises for Commutative Algebra
List 2 to hand in at 24.3.2017

Exercise 1.
Let A be a ring, S a multiplicative set in A and $f : A \to S^{-1}A$ the canonical morphism. Let $\mathcal{G} : S^{-1}A - \text{Mod} \to A - \text{Mod}$ be the "restriction of scalars functor" (Example 1.2.3 from the lecture). Show that $\mathcal{G}$ has a left-adjoint $\mathcal{F}$ which sends an A-module M to $S^{-1}M$.

Exercise 2.
Let $\mathcal{D}$ be a complete category, Γ a graph and $\mathcal{C}_\Gamma$ the free category generated by Γ.

1. Show that diagrams Δ in $\mathcal{D}$ of type Γ correspond bijectively to the objects $\mathcal{F}$ in the functor category $\text{Fun}(\mathcal{C}_\Gamma, \mathcal{D})$ such that $\lim \Delta = \lim \mathcal{F}$, where we consider $\mathcal{C}_\Gamma$ as a graph and $\mathcal{F} : \mathcal{C}_\Gamma \to \mathcal{D}$ as a diagram.

2. Show that taking the limit defines a functor $\lim : \text{Fun}(\mathcal{C}_\Gamma, \mathcal{D}) \to \mathcal{D}$.

3. Let $\delta : \mathcal{D} \to \text{Fun}(\mathcal{C}_\Gamma, \mathcal{D})$ be the *diagonal functor*, which is defined as follows: it sends an object A of $\mathcal{D}$ to the functor $\mathcal{F}_A : \mathcal{C}_\Gamma \to \mathcal{D}$ with $\mathcal{F}_A(B) = A$ and $\mathcal{F}_A(f : B \to C) = \text{id}_A$ for all objects B and morphisms f of $\mathcal{C}_\Gamma$; and it sends a morphism $f : A \to B$ in $\mathcal{D}$ to the natural transformation $\eta : \mathcal{F}_A \to \mathcal{F}_B$ with $\eta_C = f$ for all objects C in $\mathcal{C}_\Gamma$. Show that δ is left adjoint to lim.

Exercise 3.
Let $\mathcal{A}$ be an additive category with kernels and cokernels. Let $f : A \to B$ be a morphism.

1. Let $\text{im}\, f$ be the image of f and $\iota : \text{im}\, f \to B$ the canonical monomorphism. Show that there exists a morphism $\pi : A \to \text{im}\, f$ such that $f = \iota \circ \pi$.

2. Show that $\text{im}\, f$ together with ι and π satisfies the following universal property: given any object C of $\mathcal{A}$ together with a monmorphism $\iota' : C \to B$ and morphism $\pi' : A \to C$, then there exists a unique morphism $\mu : \text{im}\, f \to C$ such that $\iota = \iota' \circ \mu$ and $\pi' = \mu \circ \pi$.

3. Show that there is a canonical morphism $\text{coim}\, f \to \text{im}\, f$.

4. Conclude that every morphism $f : A \to B$ in an abelian category factors into an epimorphism $g : A \to C$, followed by a monomorphism $C \to B$.

Exercise 4.
Let $\mathcal{A}$ be an abelian category.

1. Show that for any two objects A and B of $\mathcal{A}$, the zero morphism $0 : A \to B$ is the neutral element in the abelian group $\mathrm{Hom}(A, B)$.

2. Show that the following formulas hold whenever they make sense:
   a) $h \circ (f + g) = h \circ f + h \circ g$;
   b) $(f + g) \circ h = f \circ h + g \circ h$;
   c) $f \circ (-g) = -f \circ g = (-f) \circ g$;

3. Show that a morphism $f : A \to B$ is a mono (epi) if and only if $\ker f = 0$ ($\operatorname{coker} f = 0$).

4. Show that every morphism $f : A \to B$ induces short exact sequences

$$0 \longrightarrow \ker f \longrightarrow A \longrightarrow \operatorname{coim} f \longrightarrow 0$$

   and

$$0 \longrightarrow \operatorname{im} f \longrightarrow B \longrightarrow \operatorname{coker} f \longrightarrow 0.$$

*__Exercise 5__ (Yoneda lemma)**.**
Let $\mathcal{C}$ be a small category, A an object of $\mathcal{C}$ and $h_A = \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathrm{Sets}$ and $\mathcal{F} : \mathcal{C} \to \mathrm{Sets}$ objects of $\mathrm{Fun}(\mathcal{C}, \mathrm{Sets})$. Then

$$\eta_{A,\mathcal{F}} : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Sets})}(h_A, \mathcal{F}) \longrightarrow \mathcal{F}(A)$$

is a bijection, which is functorial in A and $\mathcal{F}$.

*__Exercise 6__ (Yoneda embedding)**.**
Let $\mathcal{C}$ be a small category and $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Sets})$ the Yoneda embedding, which sends an object A to the functor $h_A = \mathrm{Hom}_{\mathcal{C}}(A, -)$ and a morphism $f : A \to B$ to the natural transformation $\eta = f_* : h_A \to h_B$, defined by $\eta_C(g) = g \circ f$ for $g \in h_A(C)$. Show that the Yoneda embedding is fully faithful.

The starred exercises are bonus exercises.