Exercises for Commutative Algebra
List 4 to hand in at 10.4.2017

Exercise 1.
Let A be a ring, $g \in A$ and $S \subset A$. Let U_g be the associated principal open subset of $\operatorname{Spec} A$. Show that $U_g \subset \bigcup_{h \in S} U_h$ if and only if g is an element of the ideal generated by S. Conclude that $\operatorname{Spec} A$ is quasi-compact.

Exercise 2.
Let A and B be rings.

1. A topological space is *irreducible* if it is non-empty and if it cannot be written as the union of two proper closed subsets. Show that $\operatorname{Spec} A$ is irreducible if and only if the nilradical $\operatorname{Nil}(A)$ of A is a prime ideal.
2. Show that $\operatorname{Spec}(A \times B)$ is homeomorphic to the disjoint union of $\operatorname{Spec} A$ with $\operatorname{Spec} B$. Conclude that Spec sends finite products to finite coproducts.

   Bonus exercise: Does Spec send infinite products to infinite coproducts?

Exercise 3.
Let k be an algebraically closed field and $X \in \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ affine k-varieties with respective rings of regular functions $A_X = k[T_1, \dots, T_n]/I_X$ and $A_Y = k[T_1, \dots, T_m]/I_Y$.

1. Let $\varphi : Y \to X$ be a regular map that is given by the rule $\varphi(\mathfrak{m}_a) = \mathfrak{m}_b$ where $b = (g_1(a), \dots, g_n(a))$ for polynomials $g_1, \dots, g_n \in k[T_1, \dots, T_m]$.
   a) Show that $\varphi^*(f) = f \circ \varphi$ defines a homomorphism $\varphi^* : A_X \to A_Y$ of k-algebras.
   b) Show that $\varphi^*([T_i]) = [g_i]$ where $[T_i]$ is the class of T_i in A_X and $[g_i]$ is the class of g_i in A_Y.
2. Let $f : A_X \to A_Y$ be a homomorphism of k-algebras and $f([T_i]) = [f_i]$ for certain $f_1, \dots, f_n \in k[T_1, \dots, T_m]$.
   a) Show that for any $a = (a_1, \dots, a_m) \in k^m$, the linear polynomial $T_i - f_i(a)$ is an element of $f^{-1}(\overline{\mathfrak{m}}_a)$.
   b) Conclude that $f^{-1}(\overline{\mathfrak{m}}_a) = \overline{\mathfrak{m}}_b$ for $b = (f_1(a), \dots, f_n(a))$ and thus $f^* : Y \to X$ is a regular map.
3. Prove Theorem 2 of section 2.3 of the lecture.

Exercise 4.
Let k be an algebraically closed field and $X \in \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ affine k-varieties with respective rings of regular functions A_X and A_Y. Let Z be the Cartesian product of X and Y (as sets), which is naturally a subset of $\mathbb{A}_k^{n+m}$.

1. Show that Z together with the inclusion $Z \subset \mathbb{A}_k^{n+m}$ is a k-variety whose ring of regular functions A_Z is isomorphic to $A_X \otimes_k A_Y$.
2. Show that Z, together with the obvious projections $\pi_X : Z \to X$ and $\pi_Y : A \to Y$, is the product of X and Y in the category of affine k-varieties.