Exercises for Commutative Algebra
List 6 to hand in at 24.4.2017

Exercise 1.
Find a minimal primary decomposition of $I = (x^3y - x^3, x^2y^2 - x^2y, xy^3 - xy^2)$ in $k[x, y]$ and determine the set $\mathrm{Ass}(I)$ of associated primes of I. Find generators of $I_\Sigma = \bigcap_{\mathfrak{q}_i \in \Sigma} \mathfrak{q}_i$ for all isolated subsets Σ of $\mathrm{Ass}(I)$. Make an illustration of the affine variety $Z(I)$ in $\mathbb{A}^2_k$.

Exercise 2.
Let A be a ring and I an irreducible ideal of A. Show that the following conditions on I are equivalent.

1. I is primary.
2. For every multiplicative set S in A and localization $\iota : A \to S^{-1}A$, we have $\iota^{-1}(S^{-1}I) = (I : a)$ for some $a \in A$.
3. The sequence
$$(I : 1) \ \subset \ (I : a) \ \subset \ (I : a^2) \ \subset \ \cdots$$
is stationary for every $a \in A$.

Show that the ideal $I = (x^2, xy, y^2)$ of $A = k[x, y]$ is primary, but not irreducible.

Exercise 3.
Let A be a ring.

1. If $A[T]$ is Noetherian, is A necessarily Noetherian?
2. If A is a subring of a Noetherian ring B, is A necessarily Noetherian?
3. If the localization $A_\mathfrak{p}$ of A at every prime ideal $\mathfrak{p}$ is Noetherian, is A necessarily Noetherian?

Exercise 4.
Let k be an algebraically closed field and $V \subset \mathbb{A}^n_k$ be an affine variety. Show that there are finitely many polynomials $f_1, \ldots, f_r \in k[T_1, \ldots, T_n]$ such that
$$V \ = \ \{\, a \in \mathbb{A}^n_k \mid f_1(a) = \ldots = f_r(a) = 0 \,\}.$$