Exercises for Commutative Algebra
List 7 to hand in at 1.5.2017

Exercise 1 (Irreducible components).
Let A be a ring and I and ideal of A. Show that the following are equivalent.

1. $V(I)$ is an irreducible topological subspace of $\operatorname{Spec} A$.
2. $\sqrt{I}$ is a prime ideal.
3. $V(I)$ contains a unique minimal prime ideal.

Assume that I has a primary decomposition and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the isolated primes of I. Show that $V(I) = \cup_{i=1}^n V(\mathfrak{p}_i)$ is the unique minimal decomposition of $V(I)$ into irreducible topological subspace of $\operatorname{Spec} A$.

Remark: If $V(I)$ is irreducible, then its unique minimal prime ideal is called its *generic point*. The subspaces $V(\mathfrak{p}_i)$ of $V(I)$ are called the *irreducible components* of $V(I)$.

Exercise 2 (Integrally closed is a local property).
Let A be an integral domain. Show that the following are equivalent.

1. A is integrally closed.
2. $A_\mathfrak{p}$ is integrally closed for every prime ideal $\mathfrak{p}$ of A.
3. $A_\mathfrak{m}$ is integrally closed for every maximal ideal $\mathfrak{m}$ of A.

Exercise 3.
Let $A \subset B$ be an integral extension of rings. Show that $A^\times = B^\times \cap A$ and that $\operatorname{Jac}(A) = \operatorname{Jac}(B) \cap A$.

Exercise 4 (Faithfully flat algebras).
Let $f : A \to B$ be a flat A-algebra. Show that the following are equivalent.

1. $f^*(f_*(I)) = I$ for all ideals I of A.
2. $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.
3. $f_*(\mathfrak{m}) \neq B$ for every maximal ideal $\mathfrak{m}$ of A.
4. $M \otimes_A B \neq \{0\}$ for every A-module $M \neq \{0\}$.
5. The A-linear map $\iota : M \to M \otimes_A B$ with $\iota(m) = m \otimes 1$ is injective for every A-module M.

Remark: A flat A-algebra with these properties is called *faithfully flat*. See Exercise 16 of Atiyah-Macdonald's book for hints.

Exercise 5 (Flat algebras have the going down property).
Let $f : A \to B$ be a flat A-algebra, $\mathfrak{q} \subset B$ a prime ideal and $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Show that $A_\mathfrak{p} \to B_\mathfrak{q}$ is faithfully flat. Conclude that $A \to B$ has the *going-down property*, i.e. for every prime ideal $\mathfrak{p}' \subset \mathfrak{p}$ of A, there is a prime ideal $\mathfrak{q}' \subset \mathfrak{q}$ of B such that $\mathfrak{p}' = f^{-1}(\mathfrak{q}')$.