Exercises for Commutative Algebra
List 9 to hand in at 22.5.2017

Exercise 1.
Let k be a field and A a finitely generated Artinian k-algebra. Show that $\dim_k A$ is finite.

Exercise 2.
Let A be a Noetherian ring. Show that A is Artinian if and only if $\operatorname{Spec} A$ is a discrete topological space.

Exercise 3.
Let A be a local domain that is not a field and assume that the maximal ideal $\mathfrak{m}$ is principal and that $\bigcap_{k\geq 1} \mathfrak{m}^k = (0)$. Show that A is a discrete valuation ring. Show that every Noetherian valuation ring that is not a field is a discrete valuation ring.

Exercise 4.
Let k be a field. Show that $A = k[X,Y]/(XY-1)$ is a Dedekind domain.

Exercise 5 (Gauß lemma for Dedekind domains)**.**
Let A be a Dedekind domain and $f = a_n T^n + \cdots + a_0 \in A[T]$. Define the *content of* f as the ideal $c(f) = (a_0, \ldots, a_n)$ of A. Show that $c(fg) = c(f)c(g)$.