Exercises for Algebra II
List 11 To hand in at 12.11.2018 in the exercise class

Exercise 1.
Consider the natural action of S_4 on $\{1,2,3,4\}$ and define H as the stabilizer of 4. Show that $H \simeq S_3$. Determine the decompositions of $\operatorname{Ind}_H^{S_4} V$ into irreducible representations for every irreducible representation V of H.

Exercise 2.
Let $G = \mathrm{SL}_2(\mathbb{F}_q)$ for some prime power q and $H = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\}$ be the subgroup of upper triangular matrices. Let $\omega : k^\times \to \mathbb{C}^*$ be a group homomorphism. Show that $\rho : H \to \mathbb{C}^\times$ with $\rho(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)) = \omega(a)$ defines a 1-dimensional complex representation V of H. Show that $\operatorname{Ind}_H^G V$ is irreducible if and only if $\omega^2 \neq 1$.

Exercise 3.
Let K be a field, G a finite group and H a subgroup of G. Fix a set of representatives $h_1, \ldots, h_r$ for G/H. For a representation V of H over K define an action of G on $K^{G/H} \otimes_K V$ by the rule $g.(h_i \otimes v) = (gh_ih^{-1}) \otimes (h.v)$ where h is the unique element in H such that $gh_ih^{-1} \in \{h_1, \ldots, h_r\}$. Show that $K^{G/H} \otimes_K V$ is a well-defined representation of G and that it is naturally isomorphic to $\operatorname{Ind}_H^G V$.

Exercise 4.
Let $G = \{\, \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_3) \mid a, b, d \in \mathbb{F}_3 \,\}$ be the subgroup of upper triangular matrices.

1. Determine all conjugacy classes of G.
2. Show that $N = \{\, \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{F}_3 \,\}$ is a normal subgroup of G and that $G^{\mathrm{ab}} = G/N$.
3. Determine all one dimensional characters of G.
4. Let X be the conjugacy class of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that G acts by conjugation on X, which defines a permutation representation $\mathbb{C}^X$. Show that $\mathbb{C}^X$ is irreducible.
5. Complete the character table of G.

***Exercise 5** (Bonus exercise)**.**
Let G be a finite group and H a subgroup of G. Let K be a field. For a representation V of H, define

$$\operatorname{Coind} V \;=\; \operatorname{Coind}_H^G V \;=\; \operatorname{Hom}_H(\operatorname{Res}_H^G K^G, V) \;\simeq\; ((K^G)^* \otimes_K V)^H$$

where K^G is the regular representation of G.

1. Show that $\operatorname{Coind} V$ is a representation of G with respect to its natural structure as a K-vector space and with the action defined by $g.\alpha(h) = \alpha(gh)$ where $\alpha : \operatorname{Res} K^G, V$ is an H-equivariant homomorphism.

2. Given an H-equivariant homomorphism $f : V \to V'$ of representations of H, show that $\alpha \mapsto f \circ \alpha$ defines a G-equivariant homomorphism $\operatorname{Coind} V \to \operatorname{Coind} V'$ of representations of G. Conclude that this defines a functor $\operatorname{Coind}_H^G : \operatorname{Rep}_K(H) \to \operatorname{Rep}_K(G)$.

3. Show that Coind_H^G is right adjoint to Res_H^G.

4. Assume that $K = \mathbb{C}$. Show that the association

$$\alpha \;\longmapsto\; \frac{1}{\#H} \sum_{g \in G} g^{-1} \otimes \alpha(g)$$

defines a canonical isomorphism $\operatorname{Coind} V \to \operatorname{Ind} V$. Conclude that Ind_H^G is left and right adjoint to Res_H^G.