

Exercises for Algebra II

List 7

To hand in at 15.10.2018 in the exercise class

Exercise 1.

Let G be a finite group, K a field and V a finite dimensional K -vector space. Let $\theta : G \times V \rightarrow V$ be a map that satisfies

$$e.v = v, \quad (gh).v = g.(h.v), \quad g.(v+w) = g.v + g.w \quad \text{and} \quad g.(\lambda v) = \lambda(g.v)$$

for all $g, h \in G$, $\lambda \in K$ and $v, w \in V$. Show that there is a unique K -linear representation $\rho : G \rightarrow \text{GL}(V)$ such that $\theta(g, v) = \rho(g)v$ for all $g \in G$ and $v \in V$.

Exercise 2.

Determine all 1-dimensional real representations of the symmetric group S_n for $n \geq 2$.

Exercise 3.

Let G be a finite group and $[G, G] = \{aba^{-1}b^{-1} \mid a, b \in G\}$ its *commutator subgroup*. Show that $[G, G]$ is a normal subgroup of G and that $G^{\text{ab}} = G/[G, G]$ is an abelian group. Show that every group homomorphism $\varphi : G \rightarrow H$ into an abelian group H factors into the quotient map $\pi : G \rightarrow G^{\text{ab}}$ followed by a uniquely determined group homomorphism $\varphi^{\text{ab}} : G^{\text{ab}} \rightarrow H$. Conclude that $[G, G] \subset \ker \rho$ for every 1-dimensional representation ρ of G .

Exercise 4.

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group with $i^2 = j^2 = k^2 = -1$ and $ij = k$. Show that there exists a unique 3-dimensional real representation $\rho : Q_8 \rightarrow \text{GL}_3(\mathbb{R})$ with

$$\rho(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho(j) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that ρ is not irreducible and determine $\ker \rho$.

Exercise 5.

Consider the natural action of S_3 on $X = \{1, 2, 3\}$ and the associated permutation representation $\rho : S_3 \rightarrow \text{GL}_3(\mathbb{C})$. Determine all subrepresentations of ρ . Which subrepresentations are irreducible?