Exercises for Algebra II
List 8 To hand in at 22.10.2018 in the exercise class

Exercise 1.
Let G be a finite group of order n and K a field whose characteristic does not divide n. Show that

$$\pi(v) \;=\; \frac{1}{n} \cdot \sum_{g \in G} g.v$$

defines a G-invariant projection $\pi : V \to V^G$.

Exercise 2.
Show that the action of S_4 on the vertices of the regular tetrahedron defines an irreducible 3-dimensional real representation V_3 of S_4. Show that the permutation representation of S_4 on $\mathbb{R}^4$, by permuting $\{1, 2, 3, 4\}$, is isomorphic to the direct sum of V_3 with the trivial 1-dimensional representation.

Exercise 3.
Show that $K[G_1 \times G_2]$ and $K[G_1] \otimes_K K[G_2]$ are isomorphic rings for finite groups G_1 and G_2. Formulate and prove a universal property for $K[G]$.

Exercise 4.
Let K be a field that contains a primitive n-th root of unity ζ_n.

1. Let G be a cyclic group of order n with generator g. Show that for every $k = 1, \ldots, n$, the map

$$\begin{array}{cccc} \pi_k : & K[G] & \longrightarrow & K \\ & \sum c_i g^i & \longmapsto & \sum c_i \zeta_n^{ki} \end{array}$$

   is a ring homomorphism.

2. Show that $\pi = (\pi_1, \ldots, \pi_n) : K[G] \to K^n$ is a ring isomorphism.

   Hint: Note that the restrictions $\chi_k : G \to K$ of π_k to G are characters in the sense of the first part of the course, which are linearly independent by Theorem 4.4.3. Why does this imply that the kernel of π is trivial?

3. Conclude that $K[G] \simeq K^n$ for every finite abelian group G of order n if $\zeta_n \in K$.

Exercise 5.
Recall the definitions of monomorphisms, epimorphisms and isomorphisms in a category. Let $f : V \to W$ be a morphism in $\mathrm{Rep}_K(G)$, i.e. a G-equivariant homomorphism. Show that

1. f is a monomorphism if and only if f is injective;
2. f is an epimorphism if and only if f is surjective;
3. f is an isomorphism if and only if f is bijective.

Show that every monomorphism in $\mathrm{Rep}_K(G)$ is a kernel and that every epimorphism in $\mathrm{Rep}_K(G)$ is a cokernel.