Matroids in tropical geometry

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What is a hyperring?

A **hyperring** is a commutative monoid $(R, \cdot, 1)$ together with a *hyperaddition*, which is a map

$$\boxplus : R \times R \longrightarrow \mathcal{P}(R) \qquad (= \text{power set of } R)$$

that satisfies for all $a, b \in R$,

1. $a \boxplus b \neq \emptyset$ (non-empty sums)
2. $a \boxplus b = b \boxplus a$ (commutative)
3. $\exists 0 \in R$ such that $0 \boxplus a = \{a\}$ (additive unit)
4. $\exists!(-a) \in R$ with $0 \in a \boxplus (-a)$ (additive inverses)
5. ... (associativity)
6. ... (distributivity)

What is a hyperfield?

A **hyperfield** is a hyperring R whose unit group

$$R^\times \;=\; \{\, a \in R \mid ab = 1 \text{ for some } b \in R \,\}$$

equals $R - \{0\}$.

Examples:

- *Viro's tropical hyperfield* $\mathbb{T}$ ...
- A field K becomes a hyperfield w.r.t. $a \boxplus b := \{a + b\}$.
- *Krasner's hyperfield* $\mathbb{K} = \{0, 1\}$ with

$\cdot$	0	1
0	0	0
1	0	1

and

$\boxplus$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$

What is a tropical variety?

Let

$$\mathbb{T}[T_1, \ldots, T_n] = \left\{ \sum_{\text{finite}} a_{\mathbf{e}} T^{\mathbf{e}} \,\middle|\, a_{\mathbf{e}} \in \mathbb{T} \right\}$$

be the set of *tropical polynomials*

$$\sum a_{\mathbf{e}} T^{\mathbf{e}} = \sum a_{(e_1,\ldots,e_n)} T_1^{e_1} \cdots T_n^{e_n}$$

Note: $\mathbb{T}[T_1, \ldots, T_n]$ carries the structure of a semiring w.r.t. the usual multiplication and the *tropical addition*

$$\sum a_{\mathbf{e}} T^{\mathbf{e}} + \sum b_{\mathbf{e}} T^{\mathbf{e}} = \sum \max\{a_{\mathbf{e}} + b_{\mathbf{e}}\} T^{\mathbf{e}}.$$

It is, however, *not extending the hyperaddition* of $\mathbb{T}$ in a natural way.

What is a tropical variety?

Let $f = \sum a_{\mathbf{e}} T^{\mathbf{e}}$ be a tropical polynomial. For a point $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$, define

$$f(x) \;=\; \sum a_{\mathbf{e}} x^{\mathbf{e}} \;=\; \sum a_{(e_1,\ldots,e_n)} x_1^{e_1} \cdots x_n^{e_n}.$$

The **corner locus of f** is the subset

$$V(f) \;=\; \Big\{ x = (x_1, \ldots, x_n) \in \mathbb{T}^n \;\Big|\; \underbrace{0 \in \boxplus\, a_{\mathbf{e}} x^{\mathbf{e}}}_{\text{i.e. the maximum is assumed twice}} \Big\}.$$

If I is an *ideal* of $\mathbb{T}[T_1, \ldots, T_n]$, which is a subset with $0 \in I$, $I + I = I$ and $I \cdot \mathbb{T}[T_1, \ldots, T_n] = I$, then define

$$V(I) \;=\; \bigcap_{f \in I} V(f).$$

Tropicalizations of classical varieties

Let K be an algebraically closed field and $v : K \to \mathbb{T}$ $(= \mathbb{R}_{\geq 0})$ a non-archimedean absolute value, aka, a morphism of hyperfields. Let X be a closed subvariety of K^n and $I \subset K[T_1, \ldots, T_n]$ the vanishing ideal. We define

$$\mathrm{Trop}(f) = \sum v(a_{\mathbf{e}})T^{\mathbf{e}} \quad \in \mathbb{T}[T_1, \ldots, T_n]$$

for $f = \sum a_{\mathbf{e}} T^{\mathbf{e}}$ in $K[T_1, \ldots, T_n]$,

$$\mathrm{Trop}(I) \;=\; \left\{ \mathrm{Trop}(f) \,\middle|\, f \in I \right\} \quad \text{and} \quad \mathrm{Trop}(X) \;=\; V\big(\mathrm{Trop}(I)\big).$$

Theorem: The map $v^n : K^n \to \mathbb{T}^n$ maps X to $\mathrm{Trop}(X)$.

Proof: If $v(a+b) < \max\big\{v(a), v(b)\big\}$, then $v(a) = v(b)$. Thus

$$\sum a_{\mathbf{e}} x^{\mathbf{e}} = 0 \qquad \text{implies} \qquad 0 \in \boxplus v(a_e) v^n(x)^{\mathbf{e}}. \qquad \square$$

Tropical linear spaces

Let I be an ideal of $\mathbb{T}[T_1, \ldots, T_n]$ that is generated by

$$I_1 \;=\; \{\, f \in I \text{ homogeneous of degree } 1 \,\}.$$

Problem:
In general, the corner locus $V(I)$ is not balanced.

Solution:
Impose matroid conditions.

What is a matroid?

Let K be a field. A ***K*-matroid** *of rank r with base set* $\underline{\mathbf{n}} = \{1, \ldots, n\}$ is a surjective linear map $\pi : K^n \to W$ to an r-dimensional K-vector space W.

A K-matroid is determined by several cryptomorphic descriptions:

- Bases: $\{ B \subset K^{\vee n} \,|\, \pi(B) \subset W$ is a basis $\}$ where $K^{\vee n} = \{$multiples of standard basis vectors $e_i \in K^n\}$.
- Linear dependent and independent sets: similar to bases.
- Circuits: nonzero vectors in $V = \ker \pi$ with a minimal set of nonzero coefficients.
- Rank functions: $r(S) = \dim(\pi(\langle S \rangle)$ for $S \subset K^{\vee n}$.
- Closure operators: $[S] = \pi^{-1}(\pi(\langle S \rangle)) \cap K^{\vee n}$ for $S \subset K^{\vee n}$.
- Dual pairs: embed W as orthogonal complement of $V = \ker \pi$.
- Plücker coordinates: $[\Delta_I] \in \mathrm{Gr}(r, n)_K$ where I runs through the set $\binom{\underline{\mathbf{n}}}{r}$ of r-subsets of $\underline{\mathbf{n}} = \{1, \ldots, n\}$.

What is a matroid?

Let $\binom{\underline{\mathbf{n}}}{r} = \{r\text{-subsets of } \underline{\mathbf{n}} = \{1, \ldots, n\}\}$.

A **Grassmann-Plücker function in $\mathbb{K}$** is a map

$$\delta : \binom{\underline{\mathbf{n}}}{r} \longrightarrow \mathbb{K},$$

that satisfies

- δ is not constant zero, (nonzero)
- $$0 \in \boxplus_{k=0}^{n} (-1)^k \ \delta(I \cup \{j_k\}) \ \delta(J - \{j_k\})$$

  for all $(r-1)$-subsets I and $(r+1)$-subsets $J = \{j_0, \ldots, j_r\}$ of $\underline{\mathbf{n}}$, where we use the convention $\delta(I) = 0$. (Plücker relations)

A **matroid** (or $\mathbb{K}$-*matroid*) is a class in

$$\{ \text{Grassmann-Plücker functions in } \mathbb{K} \} \ / \ \mathbb{K}^{\times}.$$

What is a valuated matroid?

Let $\binom{\underline{\mathbf{n}}}{r} = \{r\text{-subsets of } \underline{\mathbf{n}} = \{1, \ldots, n\}\}$.

A **Grassmann-Plücker function in** $\mathbb{T}$ is a map

$$\delta : \binom{\underline{\mathbf{n}}}{r} \longrightarrow \mathbb{T},$$

that satisfies

- δ is not constant zero, (nonzero)
- $$0 \in \boxplus_{k=0}^{n} (-1)^k \, \delta(I \cup \{j_k\}) \, \delta(J - \{j_k\})$$

  for all $(r-1)$-subsets I and $(r+1)$-subsets $J = \{j_0, \ldots, j_r\}$ of $\underline{\mathbf{n}}$, where we use the convention $\delta(I) = 0$. (Plücker relations)

A **valuated matroid** (or $\mathbb{T}$-*matroid*) is a class in

$$\{\text{ Grassmann-Plücker functions in } \mathbb{T}\ \} \,/\mathbb{T}^{\times}.$$

Back to tropical linear spaces

A **tropical linear space in $\mathbb{T}^n$** is the corner locus $V(I)$ of an ideal I of $\mathbb{T}[T_1, \ldots, T_n]$

- that is generated by its subset I_1 of linear polynomials and
- that is a valuated matroid, i.e. the support-minimal nonzero $f \in I_1$ form the circuit set of a valuated matroid.

Theorem:

(1) Let K be an algebraically closed field with non-archimedean absolute value $v : K \to \mathbb{T}$ and $X \subset K^n$ a linear subspace. Then $\mathrm{Trop}(X)$ is a tropical linear space in $\mathbb{T}^n$.

(2) Every tropical linear space is balanced.

Further developments

Baker-Norine '06: Riemann-Roch theorem for tropical curves.

This theory has a purely combinatorial formulation and proof.

Problem: Find a cohomological definition of h^0 and h^1 and an algebraic proof.

Jeff and Noah Giansiracusa '13: Tropicalizations as *$\mathbb{T}$-schemes*, using $\mathbb{F}_1$-geometry.

Maclagan-Rincón '14 & '16: (1) The tropicalization of a classical ideal is a *tropical ideal*, i.e. it satisfies matroid conditions.

(2) The $\mathbb{T}$-scheme associated with a tropical ideal is balanced.

L. '15: Generalization of G^2 using *blue schemes*, e.g. including a scheme theoretic version for Thuillier skeleta of Berkovich spaces.

Problem: Generalize tropical ideals to this more general setting.

Need: Matroids for more general objects than fields, $\mathbb{K}$, $\mathbb{T}$, ...

Generalizations of matroid theory

Dress '86, Dress-Wenzel '91, '92, . . . : Matroids for fuzzy rings, including cryptomorphic descriptions as

1. bases,
2. circuits, and
3. Grassmann-Plücker functions.

Interlude: what is a fuzzy ring?

A **fuzzy ring** is a possibly non-distributive commutative semiring R with 0 and 1 together with a proper ideal I (i.e. $0 \in I$, $I + I \subset I$, and $I \cdot R \subset I$) that satisfies:

- $\forall a \in R^\times \exists! b \in R^\times$ s.t. $a + b \in I$ (additive inverses of units)
- ... (weak forms of distributivity)
- ... (compatibility of distributivity and additive inverses)

Generalizations of matroid theory

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1. bases,
2. circuits, and
3. Grassmann-Plücker functions.

Important special class: valuated matroids.

Baker-Bowler '16: Matroids for hyperfields, including cryptomorphic descriptions as

1. circuits,
2. dual pairs, and
3. Grassmann-Plücker functions.

Giansiracusa-Jun-L. 16': The category of hyperfields occurs naturally as a full subcategory of the category of fuzzy rings, identifying the different notions of matroids.

From hyperfields to fuzzy rings to ordered blueprints

Let Hyp be the category of hyperfields.
Let Fuzz be the category of fuzzy rings.

$$\begin{array}{ccc} \text{Hyp} & \xrightarrow{\text{fully faithful}} & \text{Fuzz} \\ K & \longmapsto & (R, I) \end{array}$$

where

- $R = \mathcal{P}(K) - \emptyset$,
- $I = \{ A \subset K \mid 0 \in A \}$,
- $A \cdot B = \{ ab \mid a \in A, b \in B \}$,
- $A + B = \bigcup \{ a \boxplus b \mid a \in A, b \in B \}$.

Interlude: what is an ordered blueprint?

A **semiring** is a commutative semiring with 0 and 1.

An **ordered semiring** is a semiring R together with a partial order $\leq$ that satisfies for all $a, b, c, d \in R$ that

- $a \leq b$ and $c \leq d$ implies $a + c \leq b + d$, (additive)
- $a \leq b$ and $c \leq d$ implies $ac \leq bd$. (multiplicative)

An **(ordered) blueprint** B is an (ordered) semiring B^+ together with a multiplicative subset $B^\bullet$ that contains 0 and 1 and that generates B^+ as a semiring.

A **morphism of ordered blueprints** is a multiplicative and additive map $f : B_1^+ \to B_2^+$ that preserves 0, 1, the partial order and the multiplicative subsets.

From hyperfields to fuzzy rings to ordered blueprints

Let Hyp be the category of hyperfields.
Let Fuzz be the category of fuzzy rings.
Let OBlpr be the category of ordered blueprints.

$$\text{Hyp} \xrightarrow{\text{fully faithful}} \text{Fuzz} \xrightarrow{\text{fully faithful}} \text{OBlpr}$$

$$(R, I) \longmapsto B = (B^+, B^\bullet, \leq)$$

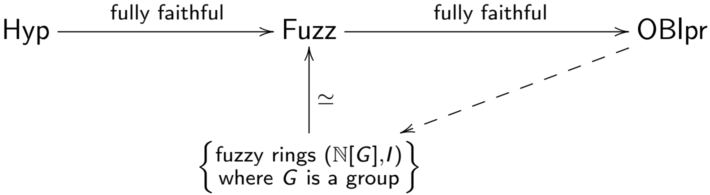
where

- $B^+ = \mathbb{N}[R^\times]$,
- $B^\bullet = R^\times \cup \{0\}$, and
- $\leq$ is generated by

$$0 \leq \sum_{\text{in } \mathbb{N}[R^\times]} a_i \qquad \text{where } a_i \in R^\times \text{ such that } \sum_{\text{in } R} a_i \in I.$$

From hyperfields to fuzzy rings to ordered blueprints

Let Hyp be the category of hyperfields.
Let Fuzz be the category of fuzzy rings.
Let OBlpr be the category of ordered blueprints.



Geometry for ordered blueprints

The **spectrum of an ordered blueprint** B is the set X of all prime ideals of B endowed with its Zariski topology and its structure sheaf $\mathcal{O}_X$ in OBlpr.

An **ordered blue scheme** is a topological space X together with a sheaf $\mathcal{O}_X$ that is locally isomorphic to the spectra of ordered blueprints.

Remark: In analogy to usual algebraic geometry, there is a Proj-construction. In particular, we can define the *projective n-space* $\mathbb{P}^n_B = \operatorname{Proj} B[T_0, \ldots, T_n]$ *over an ordered blueprint* B, as well as closed subvarieties of $\mathbb{P}^n_B$.

The moduli space of matroids

Let $B = \mathbb{F}_1^{\pm}$ be the ordered blueprint with

- $B^+ = \mathbb{N} \cdot 1 \oplus \mathbb{N} \cdot (-1)$,
- $B^\bullet = \{0, 1, -1\}$, and
- $\leq$ generated by $0 \leq 1 + (-1)$.

Remark: $\mathbb{F}_1^{\pm}$ is an initial object in the image of Fuzz $\to$ OBlpr.

Let $B_{r,n}$ be the ordered blueprint with

- $B_{r,n}^+ = \mathbb{N}\big[\, \epsilon \Delta_I \,\big|\, \epsilon \in \{\pm 1\}, I \in \binom{\mathbf{n}}{r} \,\big]$,
- $B_{r,n}^\bullet = \{0\} \cup \big\{\, \epsilon \Delta_I \,\big|\, \epsilon \in \{\pm 1\}, I \in \binom{\mathbf{n}}{r} \,\big\}$, and
- $\leq$ generated by

$$0 \;\leq\; \sum_{k=0}^{n} (-1)^k \, \Delta_{I \cup \{j_k\}} \, \Delta_{J - \{j_k\}}$$

for $(r-1)$-subsets I and $(r+1)$-subsets $J = \{j_0, \ldots, j_r\}$ of $\underline{\mathbf{n}}$.

The moduli space of matroids

Remark: Then $B_{r,n}$ is a graded $\mathbb{F}_1^{\pm}$-algebra and $\mathrm{Mat}(r,n) = \operatorname{Proj} B_{r,n}$ an ordered blue $\mathbb{F}_1^{\pm}$-scheme.

Fact: Let K be a hyperfield or fuzzy ring and B the associated ordered blueprint. Then the pullback of Plücker coordinates defines a bijection

$$\underbrace{\mathrm{Mat}(r,n)(B)}_{B\text{-rational points of } \mathrm{Mat}(r,n)} \quad \longrightarrow \; \{ \text{matroids over } K \}.$$

Wishful thinking: $\mathrm{Mat}(r,n)$ *should be* the moduli space of matroids (on $\underline{\mathbf{n}}$ of rank r).

Doubt: What about the universal family?

Families of matroids

An ordered blueprint B is **with weak inverses** if for all $a \in B^\bullet$ there is a unique $(-a) \in B^\bullet$ such that $0 \leq a + (-a)$.

An ordered blue scheme is **with weak inverses** if the values of its structure sheaf are blueprints with weak inverses.

Let X be an ordered blue scheme with weak inverses and $\mathcal{L}$ a line bundle on X. A **Grassmann-Plücker function in $\mathcal{L}$ (on $\underline{\mathbf{n}}$ of rank r)** is a map

$$\delta : \binom{\underline{\mathbf{n}}}{r} \longrightarrow \Gamma(X, \mathcal{L})$$

such that $\left\{ \delta(I) \,\middle|\, I \in M(r,n) \right\}$ generate $\mathcal{L}$ and such that

$$0 \leq \sum_{k=0}^{n} (-1)^k \delta(I \cup \{j_k\}) \delta(J - \{j_k\}) \qquad \text{in } \Gamma(X, \mathcal{L}^{\otimes 2})$$

for all $(r-1)$-subsets I and $(r+1)$-subsets J of $\underline{\mathbf{n}}$.

Families of matroids

There is a natural action of $\mathrm{Aut}(\mathcal{L})$ on the set of Grassmann-Plücker functions $\delta : M(r,n) \to \Gamma(X,\mathcal{L})$. An orbit of this action is called a **Grassmann-Plücker class in $\mathcal{L}$ (on $\underline{n}$ of rank r)**.

A **family of matroids over X (on $\underline{n}$ of rank r)** is a class $[\mathcal{L}]$ in $\mathrm{Pic}\, X$ together with a Grassmann-Plücker class $\big[\delta : M(r,n) \to \Gamma(X,\mathcal{L})\big]$ in $\mathcal{L}$.

Fact: If $X = \mathrm{Spec}\, B$ for an ordered blueprint B associated with a fuzzy ring R, then a family of matroids over B is nothing else than a matroid over R.

The moduli space of matroids

Theorem: $\mathrm{Mat}(r,n)$ is the fine moduli space of families of matroids over ordered blue schemes with weak inverses, and its universal family is the family of matroids given by the class of $\mathcal{O}(1)$ and the class of

$$\begin{array}{cccc} \delta: & \binom{\mathbf{n}}{r} & \longrightarrow & \Gamma\big(\mathrm{Mat}(r,n),\mathcal{O}(1)\big). \\ & I & \longmapsto & \Delta_I \end{array}$$

In particular, there is a natural bijection

$$\{\text{families of matroids over } X\} \longrightarrow \mathrm{Hom}\big(X,\mathrm{Mat}(r,n)\big)$$

for every ordered blue scheme X with weak inverses.