The moduli space of matroids

Oliver Lorscheid

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Based on a joint work with Matthew Baker
Theorem (Tutte ’58)

A matroid is regular if and only if it is binary and orientable.
Part 1: Pastures
Definition

A **pasture** is a pair \( F = (F^\times, N_F) \) of an abelian group \( F^\times \) (the **unit group**) and a subset \( N_F \) (the **nullset**) of the group semiring

\[
\mathbb{N}[F^\times] = \left\{ \sum_{a \in F^\times} n_a \cdot a \mid n_a \in \mathbb{N}, n_a = 0 \text{ for almost all } a \right\}
\]

that satisfies that

1. \( 0 \in N_F \);
2. \( N_F + N_F \subset N_F \);
3. \( N_F \cdot F^\times \subset N_F \);
4. there is a unique \( b \in F^\times \) for every \( a \in F^\times \) such that \( a + b \in N_F \) (the **weak inverse** of \( a \)).

The **underlying set** of \( F \) is the subset \( F^\times \cup \{0\} \) of \( \mathbb{N}[F^\times] \).

We denote the weak inverse of 1 by \( \epsilon \). We have \( \epsilon^2 = 1 \), and \( b \in F \) is the weak inverse of \( a \in F \) if and only if \( b = \epsilon a \).
Examples

1. A field \( k \) can be realized as the pasture \((k^\times, N_k)\) with

\[
N_k = \left\{ \sum n_a \cdot a \in \mathbb{N}[k^\times] \mid \sum n_a \cdot a = 0 \text{ as elements of } k \right\}.
\]

Note that the underlying set of \((k^\times, N_k)\) is \( k \) itself. For instance, 
\( \mathbb{F}_2 = (\{1\}, N_{\mathbb{F}_2}) \) with 
\( N_{\mathbb{F}_2} = \{ n \cdot 1 \mid n \in \mathbb{N} \text{ even} \} \).

2. The **Krasner hyperfield** is the pasture \( \mathbb{K} = (\{1\}, N_{\mathbb{K}}) \) with 
\( N_{\mathbb{K}} = \{ n \cdot 1 \mid n \neq 1 \} \). Note that \( \epsilon = 1 \) and \( \mathbb{K} = \{0, 1\} \) (as sets).

3. The **sign hyperfield** is the pasture \( \mathbb{S} = (\{1, \epsilon\}, N_{\mathbb{S}}) \) with
\( N_{\mathbb{S}} = \{ n_1 1 + n_\epsilon \epsilon \mid n_1 = n_\epsilon = 0 \text{ or } n_1 \neq 0 \neq n_\epsilon \} \).

4. The **regular partial field** is the pasture \( \mathbb{F}_1^\pm = (\{1, \epsilon\}, N_{\mathbb{F}_1^\pm}) \) with
\( N_{\mathbb{F}_1^\pm} = \{ n_1 1 + n_\epsilon \epsilon \mid n_1 = n_\epsilon \} \).
Morphisms

A morphism of pastures is a map \( f : F_1 \to F_2 \) between pastures \( F_1 \) and \( F_2 \) such that

1. \( f(0) = 0 \) and \( f(1) = 1 \);
2. \( f(a \cdot b) = f(a) \cdot f(b) \);
3. \( \sum n_a a \in N_{F_1} \) implies that \( \sum n_a f(a) \in N_{F_2} \).

Example

1. \( \text{sign} : \mathbb{R} \to \mathbb{S} \).
2. For every pasture \( F \), there are a unique morphism \( t_F : F \to \mathbb{K} \) (the terminal map) given by \( t_F(a) = 1 \) for all \( a \neq 0 \), and a unique morphism \( i_F : \mathbb{F}_1^\perp \to F \) given by \( i_F(0) = 0 \), \( i_F(1) = 1 \) and \( i_F(\epsilon) = \epsilon \).
Part 2: Matroids
**For the rest of the talk, let** $0 \leq r \leq n$ **be fixed integers and** $E = \{1, \ldots, n\}$. **Let** $k$ **be a field.**

**Definition**

A *$k$-matroid* (on $E$ of rank $r$) is an $r$-dimensional subspace $L$ of $k^n = \{\text{maps } E \to k\}$.

**Cryptomorphic description**

A $k$-matroid is the same as a point of the Grassmannian $\text{Gr}(r, n)(k)$, which is a subset of the projective space $\mathbb{P}^N(k) = \left\{ [\Delta_I] \mid I \in \binom{E}{r} \right\}$

where $N = \binom{n}{r} - 1$ and $\binom{E}{r}$ is the collection of all $r$-subsets of $E$. 
In other words: A $k$-matroid is a $k^\times$-class $[\Delta]$ of a \textit{Grassmann-Plücker function}, which is a nonzero map

\[
\Delta : \binom{E}{r} \longrightarrow k
\]

that satisfies the Plücker relations

\[
\sum_{k=1}^{r+1} (-1)^k \Delta_{I \cup \{j_k\}} \Delta_{J - \{j_k\}} = 0
\]

for all $I, J \subset E$ with $\#I = r - 1$, $J = \{j_1, \ldots, j_{r+1}\}$ where $j_1 < \ldots < j_{r+1}$ and $\Delta(I \cup \{j_k\}) = 0$ if $j_k \in I$. 

Let $F$ be a pasture.
An $F$-matroid is an $F^\times$-class $[\Delta]$ of a Grassmann-Plücker function, which is a nonzero map

$$\Delta : \binom{E}{r} \to F$$

that satisfies the Plücker relations

$$\sum_{k=1}^{r+1} \epsilon^k \Delta_{I \cup \{j_k\}} \Delta_{J \setminus \{j_k\}} \in N_F$$

for all $I, J \subset E$ with $\#I = r - 1$, $J = \{j_1, \ldots, j_{r+1}\}$ where $j_1 < \ldots < j_{r+1}$ and $\Delta(I \cup \{j_k\}) = 0$ if $j_k \in I$. 
Matroids

Definition
A matroid is a $\mathbb{K}$-matroid.

Example
1. Let $\Gamma$ be a connected graph, $E$ its set of edges and $r = \# E - b_1(\Gamma) = \#\{\text{edges in a spanning tree of } \Gamma\}$. Then

$$\Delta : \binom{E}{r} \rightarrow \mathbb{K}$$

$$I \mapsto \begin{cases} 1 & \text{if } I \text{ is a spanning tree of } \Gamma \\ 0 & \text{if not} \end{cases}$$

is a Grassmann-Plücker function and defines a matroid $M = [\Delta]$.

2. Let $F = k$ be a field and $r$-dimensional subspace $L \subset k^n$ with Grassmann-Plücker function $\Delta : \binom{E}{r} \rightarrow k$. Let $t_k : k \rightarrow \mathbb{K}$ be the terminal map. Then $M = [t_k \circ \Delta]$ is a matroid.
Let $F$ be a pasture with terminal map $t_F : F \to \mathbb{K}$. A matroid $M$ is representable over $F$ if there is a Grassmann-Plücker function $\Delta : \binom{E}{r} \to F$ such that $M = [t_F \circ \Delta]$.

A matroid $M$ is called

- **regular** if $M$ is representable over the regular partial field $\mathbb{F}_1^\pm$;
- **binary** if $M$ is representable over the finite field $\mathbb{F}_2$;
- **orientable** if $M$ is representable over the sign hyperfield $\mathbb{S}$.

**Theorem (Tutte, 1958)**

*A matroid is regular if and only if it is binary and orientable.*
Part 3: Moduli spaces
Ordered blueprints

An **ordered blueprint** is a triple $B = (B^\bullet, B^+, \leq)$ where

- $B^+$ is a semiring (commutative with 0 and 1);
- $B^\bullet \subset B^+$ is a multiplicatively closed subset of generators containing 0 and 1;
- $\leq$ is a partial order on $B^+$ that is **additive and multiplicative**, i.e. $x \leq y$ implies $x + z \leq y + z$ and $x \cdot z \leq y \cdot z$.

**Example**

A pasture $F = (F^\times, N_F)$ can be realized as the ordered blueprint $B = (B^\bullet, B^+, \leq)$ with

- $B^+ = \mathbb{N}[F^\times]$;
- $B^\bullet = F = F^\times \cup \{0\} \subset \mathbb{N}[F^\times]$;
- and the smallest additive and multiplicative partial order $\leq$ on $B^+$ for which $0 \leq \sum n_a a$ whenever $\sum n_a a \in N_F$. 

Ordered blue schemes

It is possible to mimic the following notions from algebra and algebraic geometry for ordered blueprints instead for rings:

- (prime) ideals;
- localizations;
- Spec $B$ and Proj $B$ (for a graded ordered blueprint $B$);
- (ordered blue) schemes;
- line bundles;
Matroid bundles

Let $X$ be an ordered blue scheme. A **Grassmann-Plücker function on** $X$ is a line bundle $\mathcal{L}$ on $X$ together with map

$$\Delta : \binom{E}{r} \rightarrow \Gamma(X, \mathcal{L})$$

that satisfies the Plücker relations and for which the family of sections $\{\Delta_I \mid I \in \binom{E}{r}\}$ generates $\mathcal{L}$.

A **matroid bundle on** $X$ is an isomorphism class $[\Delta]$ of a Grassmann-Plücker function $\Delta : \binom{E}{r} \rightarrow \Gamma(X, \mathcal{L})$.

Note that for an pasture $F$, an $F$-matroid is the same as a matroid bundle on $\text{Spec } F$.

($\text{Spec } F$ is a point and $\Gamma(\text{Spec } F, \mathcal{L}) = F$ for every line bundle $\mathcal{L}$)
The matroid space

The **matroid space** is defined as

\[
\text{Mat}(r, E) = \text{Proj}\left( F_1^\pm \left[ T_I \mid I \in \binom{E}{r} \right] \right) / (\text{Plücker relations}),
\]

which can be thought of as a Grassmannian over the regular partial field $F_1^\pm$.

It comes with an embedding $\iota : \text{Mat}(r, E) \to \mathbb{P}^N$ into projective space where $N = \binom{n}{r} - 1$.

The **universal line bundle** is $L_{\text{univ}} = \iota^* \mathcal{O}(1)$ and the **universal Grassmann-Plücker function** is

\[
\Delta_{\text{univ}} : \binom{E}{r} \rightarrow \Gamma(\text{Mat}(r, E), L_{\text{univ}}).
\]
The moduli property

**Theorem (BL18)**

Mat\((r, E)\) is the fine moduli space of matroid bundles. Its universal family is the matroid bundle \([\Delta_{\text{univ}}]\).

**Corollary**

Let \(F\) be a pasture. Then

\[
\chi : \text{Spec } F \to \text{Mat}(r, E) \quad \mapsto \quad \left\{ F\text{-matroids of rank } r \text{ on } E \right\} \quad \mapsto \quad [\chi^\# \circ \Delta_{\text{univ}}]
\]

is a bijection.
Part 4: Foundations of matroids
The universal pasture

In the case $F = \mathbb{K}$, we obtain a bijection

$$\left\{ \text{matroids of rank } r \text{ on } E \right\} \leftrightarrow \text{Mat}(r, E)(\mathbb{K}),$$

which associates with a matroid $M$ a morphism

$\chi_M : \text{Spec } \mathbb{K} \to \text{Mat}(r, E)$. Let $x_M \in \text{Mat}(r, E)$ be the image point of $\chi_M$.

The **universal pasture** of $M$ is the “residue field” $k_M$ of $\text{Mat}(r, E)$ at $x_M$. It is indeed a pasture.
First application: realization spaces

Let $M$ be a matroid and $k$ a field with terminal map $t_k : k \to \mathbb{K}$. The **realization space** of $M$ over $k$ is

$$\mathcal{X}_M(k) = \left\{ [\Delta] \in \text{Gr}(r, n)(k) \mid M = [t_k \circ \Delta] \right\}.$$  

Universality theorem (Mnev / Lafforgue / Vakil)

$\mathcal{X}_M(k)$ can be arbitrarily complicated (for fixed $k$ and varying $M$).

Theorem (BL18)

Let $k_M$ be the universal pasture of $M$. Then there exists a **canonical bijection** $\mathcal{X}_M(k) \to \text{Hom}(k_M, k)$.

Corollary

$M$ is representable over $k \iff$ there is a morphism $k_M \to k$. 
An observation

For many pastures $F$ with terminal map $t_F : F \to \mathbb{K}$ the following is true: given a matroid $M$ and a nonzero map $\Delta : \binom{E}{r} \to F$ such that $M = [t_F \circ \Delta]$, then $\Delta$ is a Grassmann-Plücker function if it satisfies the 3-term Plücker relations, i.e.

$$
\sum_{k=1}^{r+1} \epsilon^k \Delta_{I \cup \{j_k\}} \Delta_{J \setminus \{j_k\}} \in N_F
$$

for all $I, J \subset E$ with $#I = r - 1$, $J = \{j_1, \ldots, j_{r+1}\}$ where $j_1 < \ldots < j_{r+1}$ such that $#I \cap J = r - 2$.

For the purpose of this talk, we call a pasture with this property a perfect pasture.

Examples of perfect pastures are fields, $\mathbb{K}$, $\mathbb{S}$ and $F_1^\pm$. 
The foundation

The **weak universal pasture** $k^w_M$ of $M$ is defined like the universal pasture $k_M$, but only taken the 3-term Plücker relations into account.

The **foundation** of $M$ is the subpasture $k^f_M$ of $k^w_M$ that is defined by the *cross ratios*, which are terms like

$$\frac{T_{1,3} \cdot T_{2,4}}{T_{1,4} \cdot T_{2,3}}$$

(in the case $E = \{1, 2, 3, 4\}$ and $r = 2$).

**Proposition (Wenzel ’91 / BL18)**

$k^f_M$ is the “constant field” of $k^w_M$, i.e. the smallest subpasture of $k^w_M$ such that

- $(k^w_M)^\times / (k^f_M)^\times$ is a free abelian group and
- $N_{k^w_M}$ is “generated” by $N_{k^f_M}$.
Application: Tutte’s theorem

Theorem (BL18)
Let $M$ be a matroid with foundation $k_M^f$ and $F$ a perfect pasture. Then $M$ is representable over $F$ if and only if there is a morphism $k_M^f \rightarrow F$.

Theorem (BL18)
A matroid $M$ is
- regular if and only if $k_M^f = \mathbb{F}_1^\pm$;
- binary if and only if $k_M^f = \mathbb{F}_1^\pm$ or $\mathbb{F}_2$.

Theorem (Tutte ’58)
A matroid $M$ is regular if and only if it is binary and orientable.

Proof.
$M$ is binary and orientable $\iff k_M^f = \mathbb{F}_1^\pm$ or $\mathbb{F}_2$, and $M$ is orientable $\iff k_M^f = \mathbb{F}_1^\pm$ (there is no morphism $\mathbb{F}_2 \rightarrow \mathcal{S}$) $\iff M$ is regular $\square$