The moduli space of matroids

Oliver Lorscheid

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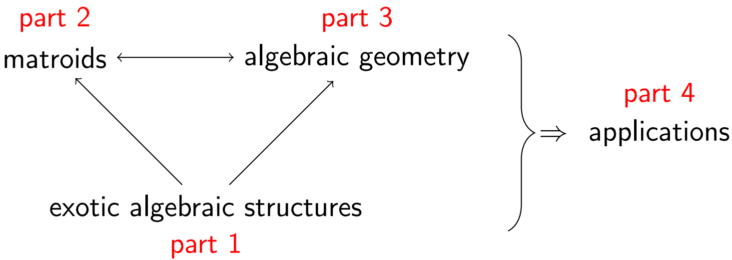
Content overview, and the thread



Theorem (Tutte '58)

A matroid is regular if and only if it is binary and orientable.

Part 1: Pastures

Definition

A **pasture** is a pair $F = (F^\times, N_F)$ of an abelian group $F^\times$ (the *unit group*) and a subset N_F (the *nullset*) of the group semiring

$$\mathbb{N}[F^\times] \;=\; \left\{ \sum_{a \in F^\times} n_a \cdot a \,\middle|\, n_a \in \mathbb{N}, n_a = 0 \text{ for almost all } a \right\}$$

that satisfies that

1. $0 \in N_F$;
2. $N_F + N_F \subset N_F$;
3. $N_F \cdot F^\times \subset N_F$;
4. there is a unique $b \in F^\times$ for every $a \in F^\times$ such that $a + b \in N_F$ (the *weak inverse* of a).

The *underlying set* of F is the subset $F^\times \cup \{0\}$ of $\mathbb{N}[F^\times]$.

We denote the weak inverse of 1 by ϵ. We have $\epsilon^2 = 1$, and $b \in F$ is the weak inverse of $a \in F$ if and only if $b = \epsilon a$.

Examples

1. A field k can be realized as the pasture $(k^\times, N_k)$ with
$$N_k \;=\; \left\{ \sum n_a \cdot a \in \mathbb{N}[k^\times] \;\middle|\; \sum n_a \cdot a = 0 \text{ as elements of } k \right\}.$$
Note that the underlying set of $(k^\times, N_k)$ is k itself.
For instance, $\mathbb{F}_2 = (\{1\}, N_{\mathbb{F}_2})$ with $N_{\mathbb{F}_2} = \{n \cdot 1 | n \in \mathbb{N} \text{ even}\}$.

2. The **Krasner hyperfield** is the pasture $\mathbb{K} = \big(\{1\}, N_{\mathbb{K}}\big)$ with $N_{\mathbb{K}} = \{n \cdot 1 | n \neq 1\}$. Note that $\epsilon = 1$ and $\mathbb{K} = \{0, 1\}$ (as sets).

3. The **sign hyperfield** is the pasture $\mathbb{S} = \big(\{1, \epsilon\}, N_{\mathbb{S}}\big)$ with $N_{\mathbb{S}} = \{n_1 1 + n_\epsilon \epsilon | n_1 = n_\epsilon = 0 \text{ or } n_1 \neq 0 \neq n_\epsilon\}$.

4. The **regular partial field** is the pasture $\mathbb{F}_1^\pm = \big(\{1, \epsilon\}, N_{\mathbb{F}_1^\pm}\big)$ with $N_{\mathbb{F}_1^\pm} = \big\{n_1 1 + n_\epsilon \epsilon | n_1 = n_\epsilon\big\}$.

Morphisms

A **morphism of pastures** is a map $f : F_1 \to F_2$ between pastures F_1 and F_2 such that

- $f(0) = 0$ and $f(1) = 1$;
- $f(a \cdot b) = f(a) \cdot f(b)$;
- $\sum n_a a \in N_{F_1}$ implies that $\sum n_a f(a) \in N_{F_2}$.

Example

1. $\text{sign} : \mathbb{R} \to \mathbb{S}$.
2. For every pasture F, there are a unique morphism
$$t_F : F \longrightarrow \mathbb{K} \qquad \text{(the } \textit{terminal map}\text{)}$$
given by $t_F(a) = 1$ for all $a \neq 0$, and a unique morphism
$$i_F : \mathbb{F}_1^{\pm} \longrightarrow F$$
given by $i_F(0) = 0$, $i_F(1) = 1$ and $i_F(\epsilon) = \epsilon$.

Part 2: Matroids

k-matroids

For the rest of the talk, let $0 \leq r \leq n$ be fixed integers and $E = \{1, \ldots, n\}$. Let k be a field.

Definition

A ***k*-matroid** (on E of rank r) is an r-dimensional subspace L of $k^n = \{\text{maps } E \to k\}$.

Cryptomorphic description

A k-matroid is the same as a point of the Grassmannian $\mathrm{Gr}(r, n)(k)$, which is a subset of the projective space

$$\mathbb{P}^N(k) \ = \ \left\{ [\Delta_I] \,\middle|\, I \in \binom{E}{r} \right\}$$

where $N = \binom{n}{r} - 1$ and $\binom{E}{r}$ is the collection of all r-subsets of E.

k-matroids

In other words:
A k-matroid is a $k^\times$-class $[\Delta]$ of a *Grassmann-Plücker function*, which is a nonzero map

$$\Delta : \binom{E}{r} \longrightarrow k$$

that satisfies the Plücker relations

$$\sum_{k=1}^{r+1} (-1)^k \Delta_{I \cup \{j_k\}} \Delta_{J - \{j_k\}} \; = \; 0$$

for all $I, J \subset E$ with $\#I = r-1$, $J = \{j_1, \ldots, j_{r+1}\}$ where $j_1 < \ldots < j_{r+1}$ and $\Delta(I \cup \{j_k\}) = 0$ if $j_k \in I$.

F-matroids

Let F be a pasture.
An F-matroid is an $F^\times$-class $[\Delta]$ of a *Grassmann-Plücker function*, which is a nonzero map

$$\Delta : \binom{E}{r} \longrightarrow F$$

that satisfies the Plücker relations

$$\sum_{k=1}^{r+1} \epsilon^k \Delta_{I \cup \{j_k\}} \Delta_{J - \{j_k\}} \ \in \ N_F$$

for all $I, J \subset E$ with $\# I = r - 1$, $J = \{j_1, \dots, j_{r+1}\}$ where $j_1 < \dots < j_{r+1}$ and $\Delta(I \cup \{j_k\}) = 0$ if $j_k \in I$.

Matroids

Definition

A **matroid** is a $\mathbb{K}$-matroid.

Example

1. Let Γ be a connected graph, E its set of edges and $r = \#E - b_1(\Gamma) = \#\{\text{edges in a spanning tree of } \Gamma\}$. Then

$$\begin{array}{rccl} \Delta : & \binom{E}{r} & \longrightarrow & \mathbb{K} \\ & I & \longmapsto & \begin{cases} 1 & \text{if } I \text{ is a spanning tree of } \Gamma \\ 0 & \text{if not} \end{cases} \end{array}$$

   is a Grassmann-Plücker function and defines a matroid $M = [\Delta]$.

2. Let $F = k$ be a field and r-dimensional subspace $L \subset k^n$ with Grassmann-Plücker function $\Delta : \binom{E}{r} \to k$. Let $t_k : k \to \mathbb{K}$ be the terminal map. Then $M = [t_k \circ \Delta]$ is a matroid.

Regular, binary and orientable matroids

Let F be a pasture with terminal map $t_F : F \to \mathbb{K}$. A matroid M is **representable over F** if there is a Grassmann-Plücker function $\Delta : \binom{E}{r} \to F$ such that $M = [t_F \circ \Delta]$.

A matroid M is called

- **regular** if M is representable over the regular partial field $\mathbb{F}_1^{\pm}$;
- **binary** if M is representable over the finite field $\mathbb{F}_2$;
- **orientable** if M is representable over the sign hyperfield $\mathbb{S}$.

Theorem (Tutte, 1958)

A matroid is regular if and only if it is binary and orientable.

Part 3: Moduli spaces

Ordered blueprints

An **ordered blueprint** is a triple $B = (B^\bullet, B^+, \leq)$ where

- B^+ is a semiring (commutative with 0 and 1);
- $B^\bullet \subset B^+$ is a multiplicatively closed subset of generators containing 0 and 1;
- $\leq$ is a partial order on B^+ that is *additive and multiplicative*, i.e. $x \leq y$ implies $x + z \leq y + z$ and $x \cdot z \leq y \cdot z$.

Example

A pasture $F = (F^\times, N_F)$ can be realized as the ordered blueprint $B = (B^\bullet, B^+, \leq)$ with

- $B^+ = \mathbb{N}[F^\times]$;
- $B^\bullet = F = F^\times \cup \{0\} \subset \mathbb{N}[F^\times]$;
- and the smallest additive and multiplicative partial order $\leq$ on B^+ for which $0 \leq \sum n_a a$ whenever $\sum n_a a \in N_F$.

Ordered blue schemes

It is possible to mimic the following notions from algebra and algebraic geometry for ordered blueprints instead for rings:

- (prime) ideals;
- localizations;
- $\operatorname{Spec} B$ and $\operatorname{Proj} B$ (for a graded ordered blueprint B);
- (ordered blue) schemes;
- line bundles;

Matroid bundles

Let X be an ordered blue scheme. A **Grassmann-Plücker function on X** is a line bundle $\mathcal{L}$ on X together with map

$$\Delta : \binom{E}{r} \longrightarrow \Gamma(X, \mathcal{L})$$

that satifies the Plücker relations and for which the family of sections $\left\{\Delta_I \middle| I \in \binom{E}{r}\right\}$ generates $\mathcal{L}$.

A **matroid bundle on X** is an isomorphism class $[\Delta]$ of a Grassmann-Plücker function $\Delta : \binom{E}{r} \to \Gamma(X, \mathcal{L})$.

Note that for an pasture F, an F-matroid is the same as a matroid bundle on $\operatorname{Spec} F$.
($\operatorname{Spec} F$ is a point and $\Gamma(\operatorname{Spec} F, \mathcal{L}) = F$ for every line bundle $\mathcal{L}$)

The matroid space

The **matroid space** is defined as

$$\text{“}\mathrm{Mat}(r,E) \;=\; \mathrm{Proj}\left(\mathbb{F}_1^{\pm}\big[T_I \mid I \in \tbinom{E}{r}\big]/(\text{Plücker relations})\right),\text{”}$$

which can be thought of as a Grassmannian over the regular partial field $\mathbb{F}_1^{\pm}$.

It comes with an embedding $\iota : \mathrm{Mat}(r,E) \to \mathbb{P}^N$ into projective space where $N = \binom{n}{r} - 1$.

The **universal line bundle** is $\mathcal{L}_{\mathrm{univ}} = \iota^*\mathcal{O}(1)$ and the **universal Grassmann-Plücker function** is

$$\begin{array}{rccc} \Delta_{\mathrm{univ}} : & \binom{E}{r} & \longrightarrow & \Gamma(\mathrm{Mat}(r,E), \mathcal{L}_{\mathrm{univ}}). \\ & I & \longmapsto & T_I \end{array}$$

The moduli property

Theorem (BL18)

Mat(r, E) *is the fine moduli space of matroid bundles. Its universal family is the matroid bundle* $[\Delta_{\mathrm{univ}}]$.

Corollary

Let F *be a pasture. Then*

$$\begin{array}{rcl} \mathrm{Mat}(r,E)(F) & \longrightarrow & \Big\{ F\text{-matroids of rank } r \text{ on } E \Big\} \\ \chi : \mathrm{Spec}\, F \to \mathrm{Mat}(r,E) & \longmapsto & [\chi^{\#} \circ \Delta_{\mathrm{univ}}] \end{array}$$

is a bijection.

Part 4: Foundations of matroids

The universal pasture

In the case $F = \mathbb{K}$, we obtain a bijection

$$\Big\{\text{matroids of rank } r \text{ on } E\Big\} \quad \longleftrightarrow \quad \text{Mat}(r, E)(\mathbb{K}),$$

which associates with a matroid M a morphism $\chi_M : \operatorname{Spec} \mathbb{K} \to \text{Mat}(r, E)$. Let $x_M \in \text{Mat}(r, E)$ be the image point of χ_M.

The **universal pasture** of M is the "residue field" k_M of $\text{Mat}(r, E)$ at x_M. It is indeed a pasture.

First application: realization spaces

Let M be a matroid and k a field with terminal map $t_k : k \to \mathbb{K}$. The **realization space** of M over k is

$$\mathcal{X}_M(k) \;=\; \Big\{ [\Delta] \in \mathrm{Gr}(r,n)(k) \,\Big|\, M = [t_k \circ \Delta] \Big\}.$$

Universality theorem (Mnev / Lafforgue / Vakil)

$\mathcal{X}_M(k)$ can be arbitrarily complicated (for fixed k and varying M).

Theorem (BL18)

Let k_M be the universal pasture of M. Then there exists a canonical bijection $\mathcal{X}_M(k) \to \mathrm{Hom}(k_M, k)$.

Corollary

M is representable over k $\Leftrightarrow$ there is a morphism $k_M \to k$.

An observation

For many pastures F with terminal map $t_F : F \to \mathbb{K}$ the following is true: given a matroid M and a nonzero map $\Delta : \binom{E}{r} \to F$ such that $M = [t_F \circ \Delta]$, then Δ is a Grassmann-Plücker function if it satisfies the *3-term Plücker relations*, i.e.

$$\sum_{k=1}^{r+1} \epsilon^k \Delta_{I \cup \{j_k\}} \Delta_{J - \{j_k\}} \in N_F$$

for all $I, J \subset E$ with $\# I = r - 1$, $J = \{j_1, \ldots, j_{r+1}\}$ where $j_1 < \ldots < j_{r+1}$ such that $\# I \cap J = r - 2$.

For the purpose of this talk, we call a pasture with this property a **perfect pasture**.

Examples of perfect pastures are fields, $\mathbb{K}$, $\mathbb{S}$ and $\mathbb{F}_1^{\pm}$.

The foundation

The **weak universal pasture** k_M^w of M is defined like the universal pasture k_M, but only taken the 3-term Plücker relations into account.

The **foundation** of M is the subpasture k_M^f of k_M^w that is defined by the *cross ratios*, which are terms like

$$\frac{T_{1,3} \cdot T_{2,4}}{T_{1,4} \cdot T_{2,3}}$$

(in the case $E = \{1, 2, 3, 4\}$ and $r = 2$).

Proposition (Wenzel '91 / BL18)

k_M^f is the "constant field" of k_M^w, i.e. the smallest subpasture of k_M^w such that

- *$(k_M^w)^\times / (k_M^f)^\times$ is a free abelian group and*
- *$N_{k_M^w}$ is "generated" by $N_{k_M^f}$.*

Application: Tutte's theorem

Theorem (BL18)

Let M be a matroid with foundation k_M^f and F a perfect pasture. Then M is representable over F if and only if there is a morphism $k_M^f \to F$.

Theorem (BL18)

A matroid M is

- *regular if and only if $k_M^f = \mathbb{F}_1^{\pm}$;*
- *binary if and only if $k_M^f = \mathbb{F}_1^{\pm}$ or $\mathbb{F}_2$.*

Theorem (Tutte '58)

A matroid M is regular if and only if it is binary and orientable.

Proof.

M is binary and orientable $\Leftrightarrow k_M^f = \mathbb{F}_1^{\pm}$ or $\mathbb{F}_2$, and M is orientable $\Leftrightarrow k_M^f = \mathbb{F}_1^{\pm}$ (there is no morphism $\mathbb{F}_2 \to \mathbb{S}$) $\Leftrightarrow$ M is regular $\square$