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Blueprints and tropical scheme theory

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Preface

These lecture notes accompany a course that I am giving in the term March–June 2018 at IMPA. I intend to add chapters accordingly to the progress of these lectures and to regularly put new versions of these notes online. To make the changes between the different versions more visible, each version will carry a distinct date on the front page. To make it possible to print these notes chapter by chapter, chapters will start on odd pages and contain a partial bibliography. To make changes in older parts of the lectures visible, each chapter carries the date of the last changes on its initial page.

Aim of these notes

In these notes, we will introduce blueprints and blue schemes and explain how this theory can be used to endow the tropicalization of a classical variety with a schematic structure.

Once the basic constructions are explained, we intend to discuss balancing conditions and connections to related theories as skeleta of Berkovich spaces, toroidal embeddings and log-structures. We put a particular emphasis on explaining open problems in this very young branch of tropical geometry.

Main references

The central references for this course are the papers [GG14] and [GG16] by Jeffrey and Noah Giansiracusa, [MR14] and [MR16] by Maclagan and Rincón, and [Lor15] by the author. There will be plenty of secondary references, which we will cite at the appropriate places.

A useful complementary source are the lecture notes [YALE17] of a series of lectures at YALE, which were given by various experts in the area and organized by Mincheva and Payne.

I am grateful for any kind of feedback that helps me to improve these notes!

Contents

Chapter 1

Why tropical scheme theory?

In this first chapter, we explain the purpose of tropical scheme theory, its main achievements as of today and some of the central question of this new branch of tropical geometry. At the end of this chapter, we give a brief outline of the previsioned structure of the rest of these notes.

1.1 Tropical varieties

In brevity, a tropical variety is a balanced polyhedral complex. In this section, we explain this definition, starting with the case of a tropical curve, which is easier to formulate than its higher dimensional analogue.

Definition 1.1.1. A *tropical curve (in $\mathbb{R}^n$)* is an embedded graph Γ in $\mathbb{R}^n$ with possibly unbounded edges together with a weight function

$$m: \operatorname{Edge}\Gamma \longrightarrow \mathbb{Z}_{>0}$$

such that all edges have rational slopes and such that the following so-called *balancing condition* is satisfied for every vertex p of Γ: for every edge e containing p, let $v_e \in \mathbb{Z}^n$ be the *primitive vector*, which is the smallest nonzero vector pointing from p in the direction of e; then

$$\sum_{p \in e} m(e) \cdot v_e \; = \; 0.$$

Example 1.1.2. In Figure 1.1, we depict a tropical curve in $\mathbb{R}^2$, explaining the balancing condition at the three vertices of the curve.

The generalization of the involved notions to higher dimensions requires some preparation and leads us to the following definitions.

Definition 1.1.3. A *halfspace in $\mathbb{R}^n$* is a subset of $\mathbb{R}^n$ of the form

$$H \; = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \,\middle|\, a_1 x_1 + \dots + a_n x_n \geqslant b \right\}$$

with $a_1, \dots, a_n, b \in \mathbb{R}$. The halfspace H is *rational* if $a_1, \dots, a_n \in \mathbb{Q}$.

Definition 1.1.4. A *(rational) polyhedron P (in $\mathbb{R}^n$)* is an intersection of finitely many (rational) halfspaces in $\mathbb{R}^n$. A *face* of a polyhedron P is a nonempty intersection of P with a halfspace H such that the boundary of H does not contain interior points of P.

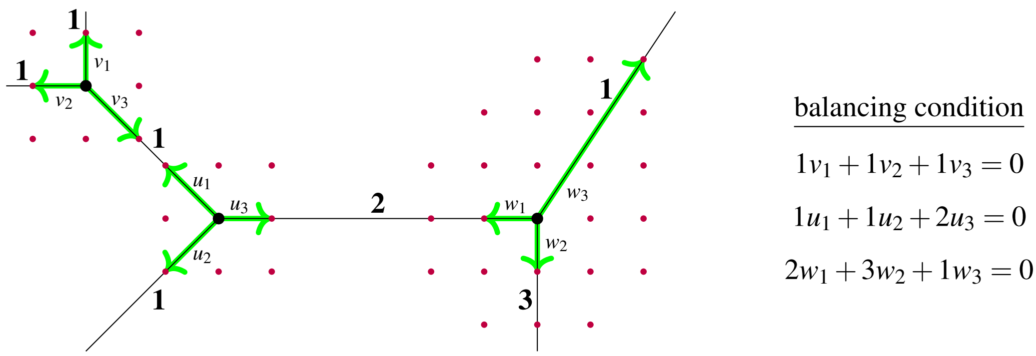

Figure 1.1: A tropical curve in $\mathbb{R}^2$ and the balancing condition

Note that the polyhedron P is a face of itself and that every face of a (rational) polyhedron is again a (rational) polyhedron.

Definition 1.1.5. A *polyhedral complex (in $\mathbb{R}^n$)* is a finite collection Δ of polyhedra in $\mathbb{R}^n$ such that the following two conditions are satisfied:

(1) each face of a polyhedron in Δ is in Δ;

(2) the intersection of two polyhedra in Δ is a face of both polyhedra or empty.

Definition 1.1.6. Let Δ be a polyhedral complex. The *support* of Δ is

$$|\Delta| = \bigcup_{P\in\Delta} P.$$

The *dimension* of Δ is $\dim\Delta = \max\{\dim P \,|\, P\in\Delta\}$. The polyhedral complex Δ is *equidimensional* if

$$|\Delta| = \bigcup_{\dim P=\dim\Delta} P$$

and Δ is *rational* if every polyhedron P in Δ is rational.

Exercise 1.1.7. Let H be a rational subvector space of $\mathbb{R}^n$, i.e. H has a basis in $\mathbb{Q}^n$. Show that the image of $\mathbb{Z}^n \subset \mathbb{R}^n$ under the quotient map $\pi : \mathbb{R}^n \to \mathbb{R}^n/H$ is a lattice, i.e. a discrete subgroup Λ that is isomorphic to $\mathbb{Z}^k$ where $k = n - \dim H$. The isomorphism $\Lambda \simeq \mathbb{Z}^k$ extends to an isomorphism $\mathbb{R}^n/H \simeq \mathbb{R}^k$ of vector spaces, i.e. we can identify π with a surjection $\pi' : \mathbb{R}^n \to \mathbb{R}^k$ that maps $\mathbb{Z}^n$ to $\mathbb{Z}^k$. Show that the image $\pi'(P)$ of a rational polyhedron P in $\mathbb{R}^n$ is a rational polyhedron in $\mathbb{R}^k$.

Let P be a rational polyhedron in $\mathbb{R}^n$ and $x_0 \in P$. Show that the subvector space H spanned by $\{x - x_0 | x \in P\}$ is rational and does not depend on the choice of x_0. Choose an isomorphism $\mathbb{R}^n/H \simeq \mathbb{R}^k$ as above. Conclude that the image $\overline{P}$ of P in $\mathbb{R}^k$ is a 0-dimensional rational polyhedron. More generally, let Q be rational polyhedron that contains P as a face. Show that the image $\overline{Q}$ of Q in $\mathbb{R}^k$ is a rational polyhedron of dimension $\dim Q - \dim P$.

We call the image $\overline{Q}$ under the quotient map $\pi' : \mathbb{R}^n \to \mathbb{R}^k$, as considered in Exercise 1.1.7, the *image of Q modulo the affine linear span of P*. If Q is a rational polyhedron of dimension $\dim Q = \dim P + 1$ that contains P as a face, then the image $\overline{Q}$ of Q in $\mathbb{R}^k$ is a one dimensional rational polyhedron that contains $\overline{P}$ as a boundary point. Thus we can speak of the *primitive vector* $v_{\overline{Q}}$ *of* $\overline{Q}$ *at* $\overline{P}$, which is the smallest nonzero vector in $\mathbb{R}^k$ with integral coefficients that is pointing from $\overline{P}$ in the direction of $\overline{Q}$.

Definition 1.1.8. A *tropical variety (in* $\mathbb{R}^n$*)* is an equidimensional and rational polyhedral complex Δ together with a weight function

$$m : \left\{ P \in \Delta \middle| \dim P = \dim \Delta \right\} \longrightarrow \mathbb{Z}_{>0}$$

such that for every polyhedron $P \in \Delta$ with $\dim P = \dim \Delta - 1$, the top dimensional polyhedra in Δ containing P satisfy the balancing modulo the affine linear span of P, i.e.

$$\sum_{P \subsetneq Q} m(Q) v_{\overline{Q}} \; = \; 0$$

where $\overline{P}$ and $\overline{Q}$ are the images of P and Q modulo the affine linear span of P and where $v_{\overline{Q}}$ is the primitive vector of $\overline{Q}$ at $\overline{P}$.

1.2 Tropicalization of classical varieties

Let k be a field.

Definition 1.2.1. A *nonarchimedean absolute value of* k is a function $v : k \to \mathbb{R}_{\geqslant 0}$ such that for all $a, b \in k$,

(1) $v(0) = 0$ and $v(1) = 1$;

(2) $v(ab) = v(a)v(b)$;

(3) $v(a+b) \leqslant \max\{v(a), v(b)\}$.

An nonarchimedean absolute value is *trivial* if $v(a) = 1$ for all $a \in k^\times$. Otherwise it is called *nontrivial*. An nonarchimedean absolute value is *discrete* if $v(k^\times)$ is a discrete subset of $\mathbb{R}_{\geqslant 0}$.

A *nonarchimedean field* is an algebraically closed field k together with a nontrivial nonarchimedean absolute value v.

Exercise 1.2.2. Let v be a nonarchimedean absolute value on a field k. Show the following assertions.

(1) If v is trivial, then v is discrete. If k is algebraically closed and v is discrete, then v is trivial. Give an example of a discrete absolute value that is not trivial. If v is not discrete, then its image in $\mathbb{R}_{\geqslant 0}$ is dense.

(2) We have $v(k^\times) \subset \mathbb{R}_{>0}$ and $v(-1) = 1$. If $v(a) \neq v(b)$, then $v(a+b) = \max\{v(a), v(b)\}$. Conclude that if $\sum_{i=1}^n a_i = 0$ in k, then at least two terms $v(a_k)$ and $v(a_l)$ with $k \neq l$ assume the maximum $\max\{v(a_i)\}$.

For the rest of this chapter, we fix a nonarchimedean field (k, v). Let $X \subset (k^\times)^n$ be an algebraic variety, i.e. the zero set of Laurent polynomials $f_1, \ldots, f_r \in k[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

Definition 1.2.3. The *tropicalization of* X is defined as the topological closure $X^{\mathrm{trop}} \; = \; \overline{\mathrm{trop}(X)}$ of the image of X under the map

$$\mathrm{trop} : (k^\times)^n \xrightarrow{(v,\ldots,v)} \mathbb{R}^n_{>0} \xrightarrow{(\log,\ldots,\log)} \mathbb{R}^n.$$

Example 1.2.4. In Figure 1.2, we illustrate the tropicalization of a genus 1 curve E, embedded sufficiently general in $(k^\times)^2$. More precisely, we illustrate the tropicalization of the compactification $\overline{E}$ of E, which embeds into the projective plane $\mathbb{P}^2$. This means that all unbounded edges of the tropicalization of E gain a second boundary point, which we illustrate by bullets in Figure 1.2. Note that this picture suggests that tropicalizations preserve certain geometric invariants like the genus.

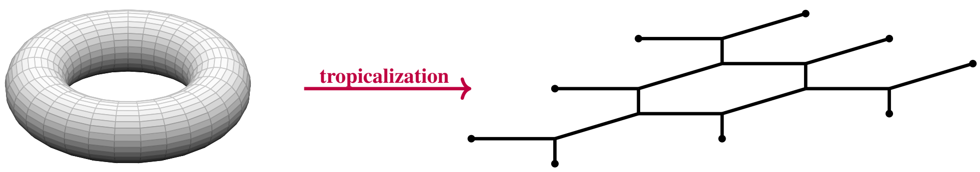

Figure 1.2: Tropicalization of an elliptic curve, including its points at infinity

Theorem 1.2.5 (Structure theorem for tropicalizations)**.** *Let (k,v) be a nonarchimedean field and $X \subset (k^\times)^n$ an equidimensional algebraic variety. Then*

(1) $X^{\text{trop}} = |\Delta|$ for a rational and equidimensional polyhedral complex Δ;

(2) $X \subset (k^\times)^n$ determines a weight function

$$m : \{P \in \Delta \,|\, \dim P = \dim \Delta\} \longrightarrow \mathbb{Z}_{>0}$$

such that (Δ, m) is a tropical variety.

The first part of the structure theorem has been proven by Bieri and Groves in their 1984 paper [BG84], which precedes tropical geometry by around 15 years and uses a slightly different setup than we do in our statement. The second part has been proven by Speyer in his thesis [Spe05]. A formulation of the structure theorem that is very close to ours appears in Maclagan and Sturmfels' book [MS15] as Theorem 3.3.6.

1.3 Two problems with the concept of a tropical variety

There are two oddities with the concept of a tropical variety that create difficulties for the development of algebro-geometric tools for tropical geometry and their application to tropicalizations of classical varieties.

The first problem is that the polyhedral complex Δ with $|\Delta| = X^{\text{trop}}$ is not determined by the classical variety $X \subset (k^\times)^n$. In other words,

the tropicalization of a classical variety is *not* a tropical variety.

The second problem relates to the functions of a tropical variety. The explanation of this issue requires some preliminary definitions.

Definition 1.3.1. The *tropical semifield* is the set $\mathbb{T} = \mathbb{R}_{\geqslant 0}$ together with the addition

$$a + b = \max\{a, b\}$$

and the usual multiplication

$$a \cdot b = ab$$

of nonnegative real numbers a, b.

Together with these operations $\mathbb{T}$ is indeed a semifield, i.e. it satisfies all of the axioms of a field except for the existence of additive inverses. The tropical semifield allows for the following reformulation of Definition 1.2.1: a nonarchimedean absolute value is a multiplicative map $v : k \to \mathbb{T}$ that is *subadditive*, i.e. $v(a+b) \leqslant v(a) + v(b)$ where the latter sum is taken with respect to the addition in $\mathbb{T}$.

Remark 1.3.2. In these lecture notes, we adopt the "max-times"-convention for the tropical numbers, which is less common than the "max-plus" or the "min-plus"-convention. To explain, the map $\log : \mathbb{T} \to \overline{\mathfrak{R}}$ defines an isomorphism of semirings between the tropical semifield $\mathbb{T}$ and the *max-plus-algebra* $\overline{\mathfrak{R}} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. Multiplication of with (-1) defines an isomorphism $\overline{\mathfrak{R}} \to \overline{\mathbb{R}}$ between the max-plus-algebra with the *min-plus-algebra* $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \min, +)$.

A priori, it is a matter of choice, with which semifield one works. But depending on the situation, some choices are more natural than others. When considering tropical varieties as polyhedral complexes, then the piecewise linear structure of the tropical variety is only visible in the logarithmic picture, i.e. one is led to work with the max-plus or the min-plus-algebra.

When working with tropical polynomials and tropical functions, in particular when compared to classical polynomials and functions, then it is more natural and less confusing to work with the max-times-convention.

Definition 1.3.3. The *tropical polynomial algebra in $T_1, \dots, T_n$* is the set

$$\mathbb{T}[T_1, \dots, T_n] \; = \; \Big\{ \sum_{J=(e_1,\dots,e_n)} a_J T_1^{e_1} \cdots T_n^{e_n} \, \Big| \, a_J \in \mathbb{T} \text{ and } a_J = 0 \text{ for almost all } J \Big\},$$

which is a semiring with respect to the usual addition and multiplication rules for polynomials where we apply the tropical addition $a_I + a_J = \max\{a_i, a_J\}$ to add coefficients.

A tropical polynomial $f = \sum a_J T_1^{e_1} \cdots T_n^{e_n}$ defines the function

$$\begin{array}{cccc} f(-): & \mathbb{T}^n & \longrightarrow & \mathbb{T}. \\ & x = (x_1, \dots, x_n) & \longmapsto & f(x) = \max\left\{ a_J x_1^{e_1} \cdots x_n^{e_n} \right\} \end{array}$$

We are prepared to explain the second problem with tropical varieties. Namely, different polynomials can define the same function, as demonstrated in the following example.

Example 1.3.4. Consider $f_1 = T^2 + 1$ and $f_2 = T^2 + T + 1$. Then

$$f_1(x) = x^2 + 1 = \max\{x^2, 1\} = \max\{x^2, x, 1\} = f_2(x)$$

for all $x \in \mathbb{T}$.

In other words,

tropical functions are *not* the same as tropical polynomials.

To understand why tropical scheme theory promises to resolve these digressions, let us have a look at classical algebraic geometry.

For varieties over an algebraically closed field, Hilbert's Nullstellensatz guarantees that functions are the same as polynomials. However, if one tries to generalize the concept of a variety to arbitrary field or even rings, one faces the same problem: different polynomials can define the same function.

Grothendieck surpassed this problem with the invention of schemes. Since the functions of a tropical variety do not form a ring, but merely a semiring, it is clear that Grothendieck's concept of a scheme does not find applications in tropical geometry.

However, $\mathbb{F}_1$-geometry has provided a theory of so-called semiring schemes, cf. the papers [Dur07] of Durov, [TV09] of Toën-Vaquié and [Lor12] of the author. This theory and its refinement in terms of blueprints provides an appropriate framework for tropical scheme theory.

1.4 Semiring schemes

In this section, we give an idea of the definition of a semiring scheme. Similar to a scheme, it is built from the spectra of semirings. In order to understand this relation between tropical varieties and semiring schemes that we have in mind, we explain this concept in analogy to classical varieties and schemes, concentrating on the affine situation. More details about the construction of semiring schemes will be explained in later parts of these notes.

Let k be an algebraically closed field and $X \subset k^n$ a variety, i.e. the zero set of polynomials $f_1, \dots, f_r \in k[T_1, \dots, T_n]$. Let

$$I \;=\; \big\{ f \in k[T_1, \dots, T_n] \,\big|\, f(x_1, \dots, x_n) = 0 \text{ for all } (x_1, \dots, x_n) \in X \big\}.$$

be its ideal of definition and $A \;=\; k[T_1, \dots, T_n]/I$ its ring of regular functions.

The associated scheme is the spectrum of A, which is the set $\operatorname{Spec} A$ of all prime ideals of A together with the topology generated by the principal open subsets

$$U_h \;=\; \{ \mathfrak{p} \subset A \,|\, h \notin \mathfrak{p} \}$$

for $h \in A$ and with the structure sheaf

$$\begin{array}{rccc} \mathcal{O}: & \{\text{open subsets of } \operatorname{Spec} A\} & \longrightarrow & \text{Rings.} \\ & U_h & \longmapsto & A[h^{-1}] \end{array}$$

We can recover the variety X from $\operatorname{Spec} A$ as follows. The ring of regular functions $A = k[T_1, \dots, T_n]/I$ equals the ring of global sections

$$\mathcal{O}(\operatorname{Spec} A) \;=\; A[1^{-1}] \;=\; A.$$

The variety X is equal to the set of k-rational points of $\operatorname{Spec} A$, i.e. we have a canonical bijection

$$X \longrightarrow \operatorname{Hom}_k(A, k) \;=\; \operatorname{Hom}_k(\operatorname{Spec} k, \operatorname{Spec} A)$$

that sends a point $x = (x_1, \dots, x_n)$ of X to the evaluation map

$$\mathrm{ev}_x : \; h \;\mapsto\; h(x).$$

Its inverse sends a homomorphism $f : A \to k$ to the point $\big(f(T_1), \dots, f(T_n)\big)$ of X.

The definition of $\operatorname{Spec} A$ extends to any semiring A as follows. There are natural extensions of the notions of prime ideals and localizations from rings to semirings.

Definition 1.4.1. The *spectrum of A* is the set $\operatorname{Spec} A$ of all prime ideals of A together with the topology generated by the principal open subsets

$$U_h = \{\mathfrak{p} \subset A \,|\, h \notin \mathfrak{p}\}$$

for $h \in A$ and with the structure sheaf

$$\begin{array}{rccc} \mathcal{O}: & \{\text{open subsets of } \operatorname{Spec} A\} & \longrightarrow & \text{Semirings} \\ & U_h & \longmapsto & A[h^{-1}] \end{array}$$

A semiring scheme is a topological space together with a sheaf in the category of semiring that is locally isomorphic to the spectra of semirings. A detailed definition of all this terminology will be given in later chapters.

1.5 Scheme theoretic tropicalization

In this section, we give an outline of the Giansiracusa tropicalization, which associates with a classical variety a semiring scheme whose $\mathbb{T}$-rational points correspond to the set theoretic tropicalization as considered in section 1.2. For the sake of simplicity, we explain this for subvarieties of affine space opposed to suvarieties of a torus, which is the context of section 1.2.

We require some notation. For a multi-index $J = (e_1, \ldots, e_n)$, we write $T^J = T_1^{e_1} \cdots T_n^{e_n}$ and $x^J = x_1^{e_1} \cdots x_n^{e_n}$. Let $f = \sum a_J T^J \in k[T_1, \ldots, T_n]$. We define

$$f^{\text{trop}} = \sum v(a_J) T^J \qquad \in \mathbb{T}[T_1, \ldots, T_n].$$

Let $X \subset k^n$ a variety with ideal of definition I.

Definition 1.5.1. The *Giansiracusa tropicalization* of X is the semiring scheme

$$\operatorname{Trop}_v(X) = \operatorname{Spec}\Big(\mathbb{T}[T_1, \ldots, T_n] \,/\, \operatorname{bend}_v(I)\Big)$$

where the *bend relations* $\operatorname{bend}_v(I)$ are defined as

$$\operatorname{bend}_v(I) = \Big(f^{\text{trop}} \sim f^{\text{trop}} + v(b_J) T^J \,\Big|\, f + b_J T^J \in I \Big).$$

The main result of Jeffrey and Noah Giansiracusa in [GG16] is the following connection to the set theoretic tropicalization X^{trop} of X, which stays in analogy to the corresponding result for schemes and varieties over an algebraically closed field.

Theorem 1.5.2 (Jeffrey and Noah Giansiracusa '13)**.** *We can recover the tropical variety X^{trop} as a set via a natural bijection*

$$X^{\text{trop}} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{T}}(\operatorname{Spec} \mathbb{T}, \operatorname{Trop}_v(X)).$$

Moreover, in case of a projective variety X, the Giansiracusa brothers associate with $\operatorname{Trop}_v(X)$ a Hilbert polynomial and show that it coincides with the Hilbert polynomial of X. This might be seen as the first striking result of tropical scheme theory.

Diane Maclagan and Felipe Rincón have shown in [MR14] that the embedding of $\operatorname{Trop}_v(X)$ into the n-dimensional tropical torus remembers the weights of the tropical variety X^{trop}, provided one has chosen the structure of a polyhedral complex. To wit, the embedding of a variety X into $(k^\times)^n$ yields an embedding of $\operatorname{Trop}_v(X)$ into the n-dimensional tropical torus $\operatorname{Spec} \mathbb{T}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

Theorem 1.5.3 (Maclagan-Rincón '14). *Assume that $X \subset (k^\times)^n$ is equidimensional. Then the weight function m of any realization of $X^{\rm trop}$ as a tropical variety (Δ, m) is determined by the embedding of $\mathrm{Trop}_v(X)$ into $\mathrm{Spec}\,\mathbb{T}[T_1^{\pm1},\dots,T_n^{\pm1}]$.*

In the author's paper [Lor15], the above results are refined and generalized by using blueprints and blue schemes. We mention two applications of this refined approach: the Giansiracusa tropicalization can be applied to more general situations than tropicalizations of subvarieties of toric varieties; for instance, it is possible to endow skeleta of Berkovich spaces with a schematic structure under certain additional hypotheses. Another feature is that the weight function of the tropical variety is already encoded into the structure sheaf of the "blue tropical scheme", which opens the possibility for a theory of abstract tropical schemes, opposed to embedded tropical schemes.

1.6 A central problem in tropical scheme theory

The aforementioned results give hope that the replacement of tropical varieties by tropical schemes will allow for new tools in tropical geometry, such as sheaf cohomology or a cohomological interpretation of intersection theory. However, it is not at all clear what a good notion of a "tropical scheme" might be.

The theory of semiring schemes comes with the notion of a $\mathbb{T}$-scheme, which is a morphism $X \to \mathrm{Spec}\,\mathbb{T}$ of semiring schemes. However, there are too many $\mathbb{T}$-schemes to make this a useful class. For example, every hyperplane in $\mathbb{R}^n$ can be realized as a $\mathbb{T}$-scheme, and such subsets of $\mathbb{R}^n$ cannot satisfy the balancing condition with respect to any polyhedral subdivision and any choice of weight function. Even worse, every intersection of hyperplanes can be realized as $\mathbb{T}$-schemes, and such intersections include all bounded convex subsets of $\mathbb{R}^n$, e.g. the unit ball.

This makes clear that we have to restrict our attention to a subclass of $\mathbb{T}$-schemes in order to obtain a useful class that could replace the class of tropical varieties. Maclagan and Rincon make a suggestion for such a class, which is based on the observation that the ideal of definition of the tropicalization of a classical variety is a valuated matroid. In [MR14] and [MR16], they investigate the class of $\mathbb{T}$-schemes whose ideal of definition is a valuated matroid and show certain desirable properties like chain conditions for "tropical ideals" and the preservation of Hilbert functions.

Unfortunately, this theory encounters some serious difficulties since the class of tropical ideals is, a priori, too restrictive. For instance, the ideals of definition of some prominent spaces in tropical geometry, like linear tropical spaces and Grassmannians, are not tropical ideals. Moreover both the intersection and the sum of two tropical ideals fail to be a tropical ideal in general, which provides obstacles for primary decompositions and intersection theory of schemes, respectively.

It might be the case that there is natural way to associate a "generically generated" tropical ideal with ideals occuring in the situations explained above, but this seems to be a difficult problem. It might be the case that the class of tropical ideals, as considered in [MR14], is too restrictive for a useful theory of "tropical schemes".

In so far, we formulate the central problem of tropical scheme theory in the following way. We would like to find a class $\mathcal{C}$ of $\mathbb{T}$-schemes that satisfies the following criteria:

- $\mathcal{C}$ contains the tropicalizations of all classical varieties and for every tropical variety, $\mathcal{C}$ contains a $\mathbb{T}$-scheme representing it;

- $\mathcal{C}$ contains "universally constructable $\mathbb{T}$-schemes" such as tropical linear spaces and tropical Grassmannians;
- the $\mathbb{T}$-rational points of every $\mathbb{T}$-scheme in $\mathcal{C}$ yields a tropical variety; in particular, this involves a theory of balancing conditions for $\mathbb{T}$-schemes;
- defining ideals of schemes in $\mathcal{C}$ are closed under intersections and sums;
- $\mathcal{C}$ allows for a dimension theory by considering chains of irreducible reduced $\mathbb{T}$-schemes in $\mathcal{C}$; in particular, this involves the notion of an irreducible $\mathbb{T}$-scheme.

A more comprehensive list of open problems in tropical scheme theory was compiled at a workshop in April 2017 at the American Institute of Mathematics, see [AIM17] for a link to the problem list.

1.7 Outline of the previsioned contents of these notes

The central goal of these notes is to explain the material of the previous sections in detail. This includes reviewing some parts of "classical" tropical geometry and introducing semiring schemes, monoid schemes and blue schemes. We intend to discuss the Giansiracusa tropicalization and subsequent results from the papers [GG14] and [GG16] by Jeffrey and Noah Giansiracusa, [MR14] and [MR16] by Maclagan and Rincón, and [Lor15] by the author.

If we achieve this central goal in time, then we intend to treat more advanced topics like scheme theoretic skeleta of Berkovich spaces, schemes over the tropical hyperfield or families of matroids.

The chapters of these notes will be grouped into parts. The first part reviews the algebraic foundations, which are (ordered) semirings, monoids, blueprints, localizations, ideals and congruences. The second part is dedicated to generalized scheme theory and contains the constructions of semiring schemes, monoid schemes and blue schemes. The third part enters the central the theme of these notes, which is scheme theoretic tropicalization.

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Part I

Algebraic foundations

Chapter 2

Semirings

In this chapter, we will provide the necessary background on semirings for our purposes. A standard source for the theory of semirings is Golan's book [Gol99], which the reader might want to confer as a secondary reference.

We illustrate the basic definitions and facts in numerous examples. Certain basic facts, which are either easy to prove or allow for a proof analogous to the case of rings, will be left as exercises.

2.1 The category of semirings

Definition 2.1.1. A *(commutative) semiring (with* 0 *and* 1*)* is a set R together with an addition $+ : R \times R \to T$, a multiplication $\cdot : R \times R \to R$ and two constants 0 and 1 such that the following axioms are satisfied:

(1) $(R,+)$ is an associative and commutative semigroup with neutral element 0;

(2) $(R,\cdot)$ is an associative and commutative semigroup with neutral element 1;

(3) $(a+b)c = ac+bc$ for all $a,b,c \in R$;

(4) $0 \cdot a = 0$ for all $a \in R$.

A *morphism between semirings* R_1 *and* R_2 is a map $f : R_1 \to R_2$ such that $f(0) = 0$, $f(1) = 1$, $f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)\cdot f(b)$ for all $a,b \in R$. We denote the category of semirings by SRings.

Let R be a semiring. A *subsemiring of* R is a subset S that contains 0 and 1 and is closed under sums and products. The *unit group* or *units of* R is the subset $R^\times$ of multiplicatively invertible elements together with the restriction of the multiplication of R to $R^\times$. A *semifield* is a semiring R such that $R^\times = R - \{0\}$.

Note that the constants 0 and 1 of a semiring R are uniquely determined as the neutral elements of addition and multiplication, respectively. In some examples, we take the liberty to omit an explicit description of these constants. Note further that the multiplication of R does indeed restrict to a multiplication $R^\times \times R^\times \to R^\times$, which turns $R^\times$ into a multiplicative group.

Remark 2.1.2. Similar to the notion of a ring, the notion of a semiring is not standardized in the literature. In other texts, the reader will find noncommutative semirings and semirings without 0 or 1. Similarly, semiring morphism might not required to preserve 0 or 1, which are properties that do not follow automatically from the other axioms. We will not encounter such weaker notions of semirings in these notes.

Example 2.1.3. Every ring is tautologically a semiring. Examples of semirings that are not rings are the following: the natural numbers $\mathbb{N}$ with respect to the usual addition and multiplication; the nonnegative real numbers $\mathbb{R}_{\geqslant 0}$ with respect to the usual addition and multiplication; and the tropical numbers $\mathbb{T}$.

Note that a subsemiring S of R is a semiring with respect to the restrictions of the addition and multiplication of R. This includes the subsemiring of *tropical integers* $\mathcal{O}_{\mathbb{T}} = \{a \in \mathbb{T} | a \leqslant 1\}$ of $\mathbb{T}$ and the subsemiring of Boolean numbers $\mathbb{B} = \{0, 1\}$ of $\mathcal{O}_{\mathbb{T}}$.

Examples of morphisms of semirings are inclusions $S \hookrightarrow R$ of subsemirings into the ambient semiring. Other examples are the following maps: $f : \mathbb{T} \to \mathbb{B}$ with $f(a) = 1$ for all $a \neq 0$; $g : \mathbb{N} \to \mathbb{B}$ with $g(a) = 1$ for all $a \neq 0$; $h : \mathcal{O}_{\mathbb{T}} \to \mathbb{B}$ with $h(a) = 0$ for all $a \neq 1$.

Exercise 2.1.4. Show that the min-plus-algebra $\overline{\mathbb{R}}$ and the max-plus-algebra $\overline{\mathfrak{R}}$, as defined in Remark 1.3.2, are semifields. What are the neutral elements for addition and multiplication? Show that the logarithm defines an isomorphism of semirings $\log : \mathbb{T} \to \overline{\mathfrak{R}}$. Show that multiplication with -1 defines an isomorphism of semirings $(-1) : \overline{\mathfrak{R}} \to \overline{\mathbb{R}}$.

Let X be a closed subset of $\mathbb{R}^n$. Show that the set $\mathrm{Fun}(X, \overline{\mathbb{R}})$ of functions from X to $\overline{\mathbb{R}}$ inherits the structure of a semiring from the addition and multiplication in $\overline{\mathbb{R}}$. Let $\mathrm{CPL}(X)$ be the smallest subring of $\mathrm{Fun}(X, \overline{\mathbb{R}})$ that contains all functions of the type $ax + b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{T}$. Show that $\mathrm{CPL}(X)$ consists of all **c**onvex **p**iecewise **l**inear functions $f : X \to \overline{\mathbb{R}}$ with integer slopes for which there is a finite covering of X by closed subsets Z_i such that $f|_{Z_i}$ is linear for each i.

Exercise 2.1.5. Let $f_1 : S \to R_1$ and $f_2 : S \to R_2$ be two morphisms of semirings. Define the tensor product $R_1 \otimes_S R_2$ as the set of finite sums $\sum a_i \otimes b_i$ of tensors $a_i \otimes b_i$ of elements $a_i \in R_1$ and $b_i \in R_2$, subject to the same relations as in the case of the tensor product of rings. Show that this forms a semiring that comes with morphisms $\iota_i : R_i \to R_1 \otimes_S R_2$ $(i = 1, 2)$, sending $a \in R_1$ to $a \otimes 1$ and $b \in R_2$ to $1 \otimes b$, respectively.

Formulate and prove the usual properties of the tensor product: (1) The tensor product is the colimit (or pushout) of the morphism f_1 and f_2; (2) every bilinear morphism from $R_1 \times R_2$ defines a unique morphism from $R_1 \otimes R_2$.

Exercise 2.1.6. Show that the category of semirings is complete and cocomplete. More precisely show the following.

(1) Show that the natural numbers $\mathbb{N}$ form an initial object and that the trivial ring $\{0 = 1\}$ forms a terminal object in SRings.

(2) Let $\{R_i\}_{i \in I}$ be a family of semirings. Then the Cartesian product $\prod_{I \in I} R_i$ together with componentwise addition and multiplication is a semiring, and the projections $\pi_j : \prod R_i \to R_j$ are semiring homomorphisms. The semiring $\prod_{I \in I} R_i$ together with the projections π_j is a product of the R_i.

(3) Let $f, g : R_1 \to R_2$ be two morphisms of semirings. Show that $\mathrm{eq}(f, g) = \{a \in R_1 | f(a) = g(a)\}$ is a subsemiring of R_1 and that the $\mathrm{eq}(f, g)$ together with the inclusion $\mathrm{eq}(f, g) \to R_1$ is an equalizer of f an g.

(4) Let $f, g : R_1 \to R_2$ be two morphisms of semirings. Show that there exists a coequalizer of f and g. *Hint:* Use Lemma 2.4.8 to show that there exists a congruence generated by the relations $f(a) \sim g(a)$ where $a \in R_1$.

(5) Let $\{R_i\}_{i\in I}$ be a finite family of semirings. Show that it has a coproduct, which we denote by $\bigotimes_{i\in I} R_i$. *Hint:* Use filtered colimits (i.e. "unions") of finite tensor products over $\mathbb{N}$.

Exercise 2.1.7. Show that a morphism $f : R_1 \to R_2$ is a monomorphism if and only if it is injective. Show that f is an isomorphism if and only if f is bijective. Show that every surjective morphism is an epimorphism. Give an example of an epimorphism that is not surjective (*hint:* cf. Exercise 2.7.3).

Exercise 2.1.8. Let $f : R \to S$ be a morphism of semirings. Show that the set theoretic image $\operatorname{im} f = f(R)$ is a subsemiring of S. Show that $\operatorname{im} f$ together with the restriction $f' : R \to \operatorname{im} f$ of f and the inclusion $\operatorname{im} f \to S$ is the categorical image of f. Conclude that every morphism factors into an epimorphism followed by a monomorphism.

2.2 First properties

We list some first properties that characterize important subclasses of semirings.

Definition 2.2.1. A semiring R is

- *without zero divisors* if for any $a,b \in R$, the equality $ab = 0$ implies that $a = 0$ or $b = 0$;
- *integral* (or *multiplicatively cancellative*) if $0 \neq 1$ and for any $a,b,c \in R$ the equality $ac = bc$ implies $c = 0$ or $a = b$;
- *strict* if $a+b = 0$ implies $a = b = 0$ for all $a,b \in R$;
- *(additively) cancellative* if for any $a,b,c \in R$ the equality $a+c = b+c$ implies $a = b$;
- *(additively) idempotent* if $1+1 = 1$.

Remark 2.2.2. While most of the above notions are standard and self-explanatory, the attribute "integral" has been used for a variety of different properties of a semiring like being without zero divisors, not being a product of two nontrivial semirings or having a unique maximal element with respect to a certain partial order.

Since "multiplicatively cancellative" seems to awkward as terminology, and its literal meaning does not indicate that $0 \neq 1$, we avoid this latter terminology in this text.

The justification for our usage of "integral" stems from the historical origin of the term "integral domain" ("Integritätsbereich" after Kronecker), which was used for generalizations of the integers to certain subrings of number fields, which are called rings of integers nowadays.[1] We will see in Exercise 2.7.4 that a semiring is integral (in our sense) if and only if it is isomorphic to a subsemiring of a semifield.

Lemma 2.2.3. *Let R be a semiring.*

(1) If $0 = 1$, then R is trivial, *i.e. R consists of the single element $0 = 1$.*

(2) If R is idempotent and cancellative, then R is trivial.

(3) If R is idempotent, then $a+a = a$ for all $a \in R$.

(4) If R is idempotent, then R is strict.

[1] For more details on the origins of "integral domain", see the answer of "t.b." in https://math.stackexchange.com/questions/45945/where-does-the-term-integral-domain-come-from#46026.

(5) *If R is integral, then R is without zero divisors.*

Proof. If $1=0$, then we have for every $a \in R$ that $a = 1\cdot a = 0\cdot a = 0$. Thus (1).

If R is idempotent and cancellative, then $1+1=1=1+0$ implies $1=0$. Thus (2).

If $1+1=1$, then we have for every $a \in R$ that $a+a = a(1+1) = a\cdot 1 = a$. Thus (3).

If R is idempotent and $a+b=0$, then we have $a = a+a+b = a+b = 0$ and similarly $b=0$. Thus (4).

If R is integral and $ab=0$, then $ab = 0 = 0\cdot b$ implies $b=0$ or $a=0$. Thus (5). □

Note that a nontrivial semiring without zero divisors does not have to be integral, in contrast to the situation for rings. An example verifying this claim is the tropical polynomial ring $\mathbb{T}[T]$, cf. Exercise 2.4.5; see Exercise 2.3.4 for another example.

Exercise 2.2.4. Verify which of the semirings from Example 2.1.3 and Exercise 2.1.4 are without zero divisors, integral, strict, cancellative or idempotent.

Exercise 2.2.5. Show that the morphism $\iota : R \to R \otimes_{\mathbb{N}} \mathbb{Z}$, sending a to $a \otimes 1$, satisfies the following properties: $R_{\mathbb{Z}} = R \otimes_{\mathbb{N}} \mathbb{Z}$ is a ring and every semiring morphism $f : R \to S$ into a ring S factors uniquely through ι. Show that R is cancellative if and only if $\iota : R \to R_{\mathbb{Z}}$ is injective. Show that R contains an *additive inverse of* 1, i.e. an element a such that $1+a=0$, if and only if R is a ring. Show that in this case $\iota : R \to R_{\mathbb{Z}}$ is an isomorphism.

2.3 Semigroup algebras and polynomial semirings

Definition 2.3.1. Let R be a semiring and A a multiplicatively written abelian semigroup with neutral element 1_A. The *semigroup algebra of A over R* is the semigroup ring $R[A]$ of finite R-linear combinations $\sum r_a a$ of elements $a \in A$, i.e. the sum contains only finitely many nonzero coefficients $r_a \in R$. The addition of $R[A]$ is defined by the formula

$$\Big[\sum r_a a\Big] + \Big[\sum s_a a\Big] \;=\; \sum (r_a + s_a) a$$

and the product is defined by the formula

$$\Big[\sum r_a a\Big] \cdot \Big[\sum s_a a\Big] \;=\; \sum_{a=bc} (r_b \cdot s_c) a.$$

The zero of $R[A]$ is the empty sum 0, i.e. the linear combination $\sum r_a a$ with $r_a = 0$ for all a, and the one of $R[A]$ is the linear combination $1 = \sum r_a a$ for which $r_{1_A} = 1$ and $r_a = 0$ for $a \neq 1_A$.

If A is the free abelian semigroup on the set of generators $\{T_i\}_{i\in I}$, then we write $R[A] = R[T_i]_{i\in I}$ or $R[A] = R[T_1,\dots,T_n]$ if $I = \{1,\dots,n\}$. We call $R[T_i]$ the *free algebra over R in $\{T_i\}$* or the *polynomial semiring over R in $\{T_i\}$*.

We allow ourselves to omit zero terms from the sums $\sum r_a a$, i.e. we may write $sb+tc$ for the element $\sum r_a a$ of $R[A]$ with $r_b = b$, $r_c = t$ and $r_a = 0$ for $a \neq b,c$. We simply write a for the element $1a$ of $R[A]$ and r for the element $r1_A$ of $R[A]$.

Exercise 2.3.2. Show that $R[A]$ is a semiring. Show that the map $\iota_R : R \to R[A]$ with $\iota_R(r) = r$ is an injective morphism of semirings. Show that the map $\iota_A : A \to R[A]$ with $\iota_A(a) = a$ is a *multiplicative map*, i.e. $\iota_A(1_A) = 1$ and $\iota_A(ab) = \iota_A(a)\cdot\iota_A(b)$ for all $a,b \in A$. Show that for every semiring morphism $f_R : R \to S$ and every multiplicative map $f_A : A \to S$, there is a unique semiring morphism $f : R[A] \to S$ such that $f_A = f \circ \iota_A$ and $f_R = f \circ \iota_R$. Use this to formulate and prove the universal property for a polynomial semiring over R.

Exercise 2.3.3. Let R be a semiring and A an abelian semigroup with neutral element. Show that $R[A] \simeq \mathbb{N}[A] \otimes_{\mathbb{N}} R$.

Exercise 2.3.4. Let $A = \{1, \epsilon\}$ be the semigroup with $\epsilon^2 = \epsilon$ and $\mathbb{B}$ the Boolean numbers (cf. Example 2.1.3). Show that $\mathbb{B}[A]$ has 4 elements. Determine the addition and multiplication table for $\mathbb{B}[A]$. Show that $\mathbb{B}[A]$ is without zero divisors, but not integral.

2.4 Quotients and congruences

Definition 2.4.1. Let R be a semiring. A *congruence on* R is an equivalence relation $\mathfrak{c}$ on R that is *additive* and *multiplicative*, i.e. (a,b) and (c,d) in $\mathfrak{c}$ imply $(a+c, b+d)$ and (ac, bd) in $\mathfrak{c}$ for all $a,b,c,d \in R$.

Exercise 2.4.2. Let R be a ring. Show that for every ideal I of R, the set $\{(a,b) | a-b \in I\}$ is a congruence on R and that every congruence is of this form.

Exercise 2.4.3. Let $k, n \in \mathbb{N}$. Show that the set

$$\mathfrak{c}_{k,n} \ = \ \left\{ (m+rk, m+sk) \in \mathbb{N}\times\mathbb{N} \,\middle|\, m,r,s \in \mathbb{N} \text{ and } m \geqslant n \text{ or } r=s=0 \right\}$$

is a congruence on $\mathbb{N}$ and that every congruence of $\mathbb{N}$ is of this form.

Given a congruence $\mathfrak{c}$ on R, we often write $a \sim_{\mathfrak{c}} b$, or simply $a \sim b$, if there is no danger of confusion, to express that (a,b) is an element of $\mathfrak{c}$. The following proposition shows that congruences define quotients of semirings.

Proposition 2.4.4. *Let R be a semiring and $\mathfrak{c}$ be a congruence. Then the associations $[a]+[b] = [a+b]$ and $[a]\cdot[b] = [ab]$ are well-defined on equivalence classes $[a]$ of $\mathfrak{c}$ and turn the quotient $R/\mathfrak{c}$ into a semiring with zero $[0]$ and one $[1]$.*

The quotient map $\pi : R \to R/\mathfrak{c}$ is a morphism of semirings that satisfies the following universal property: every morphism $f : R \to S$ of semiring such that $f(a) = f(b)$ whenever $a \sim b$ in $\mathfrak{c}$ factors uniquely through π.

Proof. Consider $a \sim a'$ and $b \sim b'$. Then $a+b \sim a'+b'$ and $ab \sim a'b'$. Thus the addition and multiplication of $R/\mathfrak{c}$ does not depend on the choice of representative and is therefore well-defined. The properties of a semiring follow immediately, including the characterization of the zero as $[0]$ and the one as $[1]$. That $\pi : R \to R/\mathfrak{c}$ is a semiring homomorphism is tautological by the definition of $R/\mathfrak{c}$.

Let $f : R \to S$ be a semiring morphism such that $f(a) = f(b)$ whenever $a \sim b$ in $\mathfrak{c}$. For f to factor into $\overline{f} \circ \pi$ for a semiring morphism $\overline{f} : R/\mathfrak{c} \to S$, it is necessary that $\overline{f}([a]) = \overline{f} \circ \pi(a) = f(a)$. This shows that $\overline{f}$ is unique if it exists. Since $a \sim b$ implies $f(a) = f(b)$, we conclude that $\overline{f}$ is well-defined as a map. The verification of the axioms of a semiring morphism are left as an exercise. □

Exercise 2.4.5. Let $n \geqslant 1$. Show that $R = \mathbb{T}[T_1, \dots, T_n]$ is without zero divisors, but not integral. Show that the relation $\{(f,g) \in R \times R | f(x) = g(x) \text{ for all } x \in \mathbb{T}^n\}$ is a congruence on R; cf. section 1.3 for definition of $f(x)$. Show that the quotient $R/\mathfrak{c}$ is integral and isomorphic to $\mathrm{CPL}(\mathbb{R}^n)$; cf. Exercise 2.1.4 for the definition of $\mathrm{CPL}(\mathbb{R}^n)$.

Conversely, every quotient is characterized by a congruence. More precisely, for every semiring morphism, there is a congruence that characterizes which elements in the domain become identified in the image.

Definition 2.4.6. Let $f : R \to S$ be a morphism of semirings. The *congruence kernel of f* is the relation $\mathfrak{c}(f) = \{(a,b) \in R \times R | f(a) = f(b)\}$ on R.

Lemma 2.4.7. *The congruence kernel $\mathfrak{c}(f)$ of a morphism $f : R \to S$ of semirings is a congruence on R.*

Proof. That $\mathfrak{c} = \mathfrak{c}(f)$ is an equivalence relation follows from the following calculations: $f(a) = f(a)$ (reflexive); $f(a) = f(b)$ implies $f(b) = f(a)$ (symmetry); $f(a) = f(b)$ and $f(b) = f(c)$ imply $f(a) = f(c)$ (transitive). Additivity and multiplicativity follow from: $f(a) = f(b)$ and $f(c) = f(d)$ imply $f(a+c) = f(a) + f(c) = f(b) + f(d) = f(b+d)$ and $f(ac) = f(a) \cdot f(c) = f(b) \cdot f(d) = f(bd)$. This shows that $\mathfrak{c}$ is a congruence. □

As a consequence of this lemma, we see that for a semiring R, the associations

$$\begin{array}{ccc} \{\text{congruences on } R\} & \longleftrightarrow & \{\text{quotients of } R\} \\ \mathfrak{c} & \longmapsto & R \to R/\mathfrak{c} \\ \mathfrak{c}(\pi) & \longleftarrow\!\shortmid & \pi : R \twoheadrightarrow R' \end{array}$$

are mutually inverse bijections. Note that strictly speaking a quotient of R is an equivalence class of surjective semiring morphisms $R \to R'$ where two surjections $\pi_1 : R \to R_1$ and $\pi_2 : R \to R_2$ are equivalent if there exists an isomorphism $f : R_1 \to R_2$ such that $f \circ \pi_2 = \pi_2$.

We will see in section 2.5 that we do not have a correspondence between quotients and ideals, as in the case of rings. In so far, one has to work with congruences when one wants to describe quotients of semirings.

A subset S of $R \times R$ is *symmetric* if $(y,x) \in S$ if $(x,y) \in S$. To make the notation "$\langle S \rangle$" compatible with the corresponding notation in section 5.2, we will consider only symmetric subsets in the following.

Lemma 2.4.8. *Let R be a semiring and $S \subset R \times R$ a symmetric subset. Then there is a smallest congruence $\mathfrak{c} = \langle S \rangle$ containing S. The quotient map $\pi : R \to R/\langle S \rangle$ satisfies the following universal property: every morphisms $f : R \to R'$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$ factors uniquely through π.*

Proof. It is readily verified that the intersection of congruences is again a congruence. As a consequence, the intersection of all congruences containing S is the smallest congruence containing S.

Given any morphism $f : R \to R'$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$, then the congruence kernel $\mathfrak{c}(f)$ must contain S and thus $\mathfrak{c} = \langle S \rangle$. Using Proposition 2.4.4, we see that f factors uniquely through π. □

This lemma shows that we can construct new semirings from known ones by prescribing a number of relations: let R be a semiring and $\{a_i \sim b_i\}$ a symmetric set of relations on R, i.e. $S = \{(a_i, b_i)\}$ is a symmetric subset of $R \times R$. Then we define $R/\langle a_i \sim b_i \rangle$ as the quotient semiring $R/\langle S \rangle$.

Exercise 2.4.9. Show that $\mathbb{B}[T]/\langle T^2 \sim T \rangle$ is isomorphic to the semigroup algebra $\mathbb{B}[A]$ where $A = \{1, \epsilon\}$ is the semigroup with $\epsilon = \epsilon^2$; cf. Exercise 2.3.4. Determine all congruences on $\mathbb{B}[A]$.

Exercise 2.4.10. Let R be a semirings and $\mathfrak{c}$ a congruence on R. Show that $\mathfrak{c}$ is a subsemiring of $R \times R$ containing the image of the diagonal map $\Delta : R \to R \times R$.

Let $f : R \to S$ be a homomorphism of semirings. Show that the congruence kernel of f together with the inclusion into $R \times R$ is the equalizer of the morphisms $f \circ \mathrm{pr}_1$ and $f \circ \mathrm{pr}_2$ from $R \times R$ to S where $\mathrm{pr}_i : R \times R \to R$ is the i-th canonical projection ($i = 1, 2$).

2.5 Ideals

While the concept of congruences is the correct generalization of ideals from rings to semirings that characterizes quotients of semirings, there are other more straight-forward generalizations of ideals, which carry over other properties from rings to semirings. In this section, we will examine two such notions: ideals and k-ideals.

Definition 2.5.1. Let R be a semiring. An *ideal of* R is a subset I of R such that 0, ac and $a+b$ are elements of I for all $a, b \in I$ and $c \in R$. A *k-ideal* or a *subtractive ideal of* R is an ideal I of R such that $a + c = b$ with $a, b \in I$ and $c \in R$ implies $c \in I$.

Once we make sense of the concept of a (semi)module over R, we could characterize an ideal of R as a submodule of R. The relevance of (prime) ideals of semirings lies in the fact that they are the good notion of points of the spectrum of R. We will come back to this in the chapter on semiring schemes.

The relevance of k-ideals is easier to explain. Namely, they form the class of subsets that is characterized as the 0-fibres, or kernels, of semiring morphisms. We assume, without evidence, that the “k” in “k-ideal” stands for “kernel”. The name k-ideal seems to be coined by Henriksen in [Hen58].

Definition 2.5.2. Let $f : R \to S$ be a semiring morphism. The *(ideal) kernel of* f is the inverse image $\ker(f) = f^{-1}(0)$ of 0.

Let S be a subset of R. The *congruence generated by* S is the congruence $\mathfrak{c}(S)$ generated by $\{(a,0) | a \in S\}$.

Proposition 2.5.3. *The kernel* $\ker(f)$ *of a morphism of semiring* $f : R \to S$ *is a k-ideal and every k-ideal appears as a kernel. More precisely, if* I *is an ideal of* R *and* $\mathfrak{c} = \mathfrak{c}(I)$ *is the congruence generated by* I*, then* $a \sim_{\mathfrak{c}} b$ *if and only if there are elements* $c, d \in I$ *such that* $a + c = b + d$*. The ideal* I *is an k-ideal if and only if* I *is the kernel of* $\pi : R \to R/\mathfrak{c}$*.*

Proof. We begin with the verification that $\ker(f)$ is a k-ideal. Clearly $0 \in \ker(f)$. Let $a, b \in \ker(f)$ and $c \in R$. Then $f(ac) = f(a)f(c) = 0 \cdot f(c) = 0$ and $f(a+b) = f(a) + f(b) = 0$, thus ac and $a+b$ are in $\ker(f)$. If $a + c = b$, then $f(c) = 0 + f(c) = f(a) + f(c) = f(b) = 0$ shows that $c \in \ker(f)$. Thus $\ker(f)$ is a k-ideal.

In order to verify the second claim of the proposition, we begin with showing that the relation

$$\mathfrak{c}' \;=\; \big\{ (a,b) \in R \times R \,|\, a + c = b + d \text{ for some } c, d \in I \big\}$$

is a congruence. Reflexivity and symmetry are immediate from the definition. Transitivity is shown as follows: if $a \sim_{\mathfrak{c}'} b \sim_{\mathfrak{c}'} b'$, then there are elements $c, d, c', d' \in I$ such that $a + c = b + d$ and $b + c' = b' + d'$. Adding c' to the former and d to the latter equation yields $a + c + c' = b + d + c' = b' + d + d'$. Since I is closed under sums, $c + c'$ and $d + d'$ are in I and thus $a \sim_{\mathfrak{c}'} b'$. This shows that $\mathfrak{c}'$ is an equivalence relation.

We continue with the verification of additivity and multiplicativity of $\mathfrak{c}'$. Let $a \sim_{\mathfrak{c}'} b$ and $a' \sim_{\mathfrak{c}'} b'$, i.e. $a+c = b+d$ and $a'+c' = b'+d'$ for some $c,d,c',d' \in I$. Adding these equations yields $a+a'+c+c' = b+b'+d+d'$ where $c+c'$ and $d+d'$ are in I. Thus $a+a' \sim_{\mathfrak{c}'} b+b'$, which establishes additivity. Multiplying these equations yields

$$aa'+ac'+a'c+cc' \;=\; (a+c)(a'+c') \;=\; (b+d)(b'+d') \;=\; bb'+bd'+b'd+dd'.$$

Since $ac'+a'c+cc'$ and $bd'+b'd+dd'$ are in I, we have $aa' \sim_{\mathfrak{c}'} bb'$, which shows multiplicativity of $\mathfrak{c}'$. Thus $\mathfrak{c}'$ is a congruence.

As the next step, we verify that $\mathfrak{c}'$ is equal to the congruence $\mathfrak{c}$ generated by I. Since $a+0 = 0+0$, we see that $\mathfrak{c}'$ contains the generating set $\{(a,0)|a \in I\}$ of $\mathfrak{c}$. Thus $\mathfrak{c}$ is contained in $\mathfrak{c}'$. Conversely, consider a relation $a \sim_{\mathfrak{c}'} b$ in $\mathfrak{c}'$, i.e. $a+c = b+d$ for some $b,d \in I$. Then $b \sim_{\mathfrak{c}} 0 \sim_{\mathfrak{c}} d$ and, by the additivity of $\mathfrak{c}$,

$$a \;=\; a+0 \;\sim_{\mathfrak{c}}\; a+c \;=\; b+d \;\sim_{\mathfrak{c}}\; b+0 \;=b,$$

i.e. $a \sim_{\mathfrak{c}} b$ in $\mathfrak{c}$. This shows that $\mathfrak{c} = \mathfrak{c}'$, as claimed.

Finally, we show that I is a k-ideal if and only if it is the kernel of $\pi : R \to R/\mathfrak{c}$, i.e. $I = \{a \in R | a \sim_{\mathfrak{c}} 0\}$. By the definition of $\mathfrak{c} = \mathfrak{c}(I)$, it is clear that $I \subset \ker(\pi)$. By the characterization of $\mathfrak{c}$ as $\mathfrak{c}'$, we have $a \in \ker(\pi)$ if and only if there are elements $c,d \in I$ such that $a+c = 0+d = d$. Thus $I = \ker(\pi)$ if and only if I is a k-ideal. This finishes the proof of the proposition. $\square$

As a consequence, proven in Corollary 2.5.4 below, we see that for every subset S of a semiring R, there is a unique smallest (k-)ideal containing S. We call this (k-)ideal the *(k-)ideal generated by S* and denote ideal generated by S by $\langle S \rangle$ and the k-ideal generated by S by $\langle S \rangle_k$.

Corollary 2.5.4. *Let R be a semiring and S a subset of R. The ideal generated by S is*

$$\langle S \rangle \;=\; \left\{ \textstyle\sum a_i s_i \,\middle|\, a_i \in R, s_i \in S \cup \{0\} \right\}.$$

The k-ideal generated by S is

$$\langle S \rangle_k \;=\; \left\{ c \in R \,\middle|\, \textstyle\sum a_i s_i + c = \sum b_j t_j \text{ for some } a_i, b_j \in R, s_i, t_j \in S \cup \{0\} \right\}.$$

Proof. Let $I = \{\sum a_i s_i | a_i \in R, s_i \in S \cup \{0\}\}$. It is clear that $S \subset I \subset \langle S \rangle$. It follows that $\langle S \rangle = I$ if we can show that I is an ideal. This can be shown directly. Clearly, $0 \in I$ and I is closed under addition. Given an element $a = \sum a_i s_i$ in I and $b \in R$, then $ab = \sum (a_i b) s_i$ is in I. This shows that I is an ideal and proves the first claim of the corollary.

Note that the right hand side of the last equation of corollary is equal to $J = \{c \in R | a+c = b \text{ for some } a,b \in I\}$ where I is as above. Let $\mathfrak{c} = \mathfrak{c}(I)$ be the congruence generated by I and $\pi : R \to R/\mathfrak{c}$ the quotient map. It follows from Proposition 2.5.3 that J is the kernel of π and thus a k-ideal. Since obviously $S \subset J \subset \langle S \rangle_k$, we conclude that $\langle S \rangle_k = J$. This completes the proof of the corollary. $\square$

To conclude, ideals, k-ideals and congruences are different generalizations of ideals to semirings, which do not coincide in general. There are ways to pass from one class to the other, which follows from our previous results.

Namely, with a congruence $\mathfrak{c}$ on a semiring R, we can associate the kernel of the projection $\pi_{\mathfrak{c}} : R \to R/\mathfrak{c}$, which is a k-ideal; with a k-ideal I, we can associate the congruence $\mathfrak{c}(I)$ generated by I. We have that the kernel of $R \to R/\mathfrak{c}(I)$ is I and the congruence $\mathfrak{c}(\ker \pi_{\mathfrak{c}})$ is contained in $\mathfrak{c}$, but in general not equal to $\mathfrak{c}$.

On the other end, every k-ideal is tautologically an ideal. With an ideal I of R, we can associate the smallest k-ideal containing I, which is the kernel of $R \to R/\mathfrak{c}(I)$. We summarize this discussion in the following picture.

$$\begin{array}{ccccc} \textbf{“submodules”} & & \textbf{“kernels”} & & \textbf{“quotients”} \\ \{\, \text{ideals of } R \,\} & \substack{\longleftarrow \\ \twoheadrightarrow} & \{\, k\text{-ideals of } R \,\} & \substack{\twoheadleftarrow \\ \longrightarrow} & \{\, \text{congruences on } R \,\} \end{array}$$

Exercise 2.5.5. Describe all ideals, k-ideals and congruences for $\mathbb{N}$; cf. Exercise 2.4.3. Describe the maps from the above diagram in this example.

Exercise 2.5.6. Let $A = \{1, \epsilon\}$ be the semigroup with $\epsilon^2 = \epsilon$ and $R = \mathbb{B}[A]$ the semigroup algebra, which has been already the protagonist of Exercises 2.3.4 and 2.4.9. Determine all ideals, k-ideals and congruences of $\mathbb{B}[A]$ and describe the above maps between ideals, k-ideals and congruences explicitly.

Exercise 2.5.7. Let R be an idempotent semiring, I an ideal of R and $\mathfrak{c} = \mathfrak{c}(I)$ the associated congruence. Show that $a \sim_{\mathfrak{c}} b$ if and only if $a + c = b + c$ for some $c \in I$. Conclude that I is a k-ideal if and only if $a + c = c$ with $c \in I$ implies $a \in I$.

Exercise 2.5.8. Let R be a cancellative semiring and I an ideal of R. Let $R_{\mathbb{Z}} = R \otimes_{\mathbb{N}} \mathbb{Z}$ and $\iota : R \to R_{\mathbb{Z}}$ be the morphism that sends a to $a \otimes 1$. Let $J = \langle \iota(I) \rangle$ be the ideal of $R_{\mathbb{Z}}$ generated by $\iota(I)$. Show that I is a k-ideal if and only if $I = \iota^{-1}(J)$. Find an example of a non-cancellative semiring R with k-ideal I such that I is not equal to $\iota^{-1}\big(\langle \iota(I) \rangle\big)$.

2.6 Prime ideals

In the last two sections of this chapter, we turn to topics of relevance for scheme theory, which are prime ideals and localizations, respectively.

Definition 2.6.1. A (k-)ideal I of R is *proper* if it is not equal to R. It is *maximal* if it is proper and if $I \subset J$ implies $I = J$ for any other proper (k-)ideal. It is *prime* if its complement $S = R - I$ is a multiplicative subset of R.

Note that a k-ideal I is a prime k-ideal if and only if it is a prime ideal. In so far, we can use the attribute “prime” unambiguously for ideals and k-ideals. Note, however, that the k-ideal generated by a prime ideal does not need to be prime; we provide proof in Example 2.6.2 below.

The situation for maximal (k-)ideals is more subtle. A k-ideal that is a maximal ideal is tautologically a maximal k-ideal. But the converse fails to be true in general, as demonstrated in Example 2.6.2. This means that we have to make a clear distinction between maximal ideals and maximal k-ideals.

Example 2.6.2. Consider the semiring $R = \mathbb{B}[T]/\langle T^2 \sim T \sim T + 1 \rangle$, which is a quotient of the semiring $\mathbb{B}[A]$ from Exercises 2.3.4, 2.4.9 and 2.5.6. It consists of the elements $0, 1, T$ and its unit group is $R^{\times} = \{1\}$. The proper ideals of R are $(0) = \{0\}$ and $(T) = \{0, T\}$, which are both prime ideals, but only (0) is a k-ideal.

This example demonstrates the following effects:

- (0) is a maximal k-ideal, but it is not a maximal ideal since it is properly contained in the proper ideal (T).

- (T) is a prime ideal, but the k-ideal generated by (T), which is R, is not a prime k-ideal.
- The quotient $R/(0)$ of R by the maximal k-ideal (0), which is equal to R, is not a semifield.
- The quotient $R/(T)$ of R by the k-ideal generated by (T), which is the trivial semiring $R/R = \{0\}$, is not a semifield.

Being warned that (k-)ideals for semirings fail to satisfy certain properties that we are used to from ideal theory of rings, we begin with the proof of properties that extend to the realm of semirings.

Lemma 2.6.3. *Let R be a semiring and I a k-ideal of R. Then I is prime if and only if R/I is nontrivial and without zero divisors.*

Proof. The k-ideal I is prime if and only if for all $a,b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$. Passing to the quotient R/I, this means that $[ab] = [0]$ implies $[a] = [0]$ or $[b] = [0]$ where we use that the kernel of $R \to R/I$ is I, cf. Proposition 2.5.3. This latter condition is equivalent to R/I being nontrivial and without zero divisors. □

Remark 2.6.4. As shown in Example 2.6.2, the usual characterization of maximal ideals as those ideals whose quotient is a field does not hold for semirings. We can only give the following quite tautological characterization of maximal k-ideals: a k-ideal I is maximal if and only if the zero ideal $\{0\}$ of R/I is a maximal k-ideal.

Lemma 2.6.5. *Every maximal ideal of a semiring is a prime ideal.*

Proof. Let R be a semiring and $\mathfrak{m}$ a maximal ideal. Consider $a,b \in R$ such that $ab \in \mathfrak{m}$, but $a \notin \mathfrak{m}$. We want to show that $b \in \mathfrak{m}$.

Since $\mathfrak{m}$ is maximal and does not contain a, the set $S = \mathfrak{m} \cup \{a\}$ generates the ideal $(1) = R$. By Corollary 2.5.4, this means that $1 = \sum e_k c_k$ for some $c_k \in S$ and $e_k \in R$. Note that $bd_k \in \mathfrak{m}$ since either $d_k \in \mathfrak{m}$ or $d_k = a$. Thus $be_k d_k \in \mathfrak{m}$ and $b = b \cdot 1 = \sum be_k c_k$is an element of $\mathfrak{m}$ as claimed, which completes the proof. □

Lemma 2.6.6. *Every maximal k-ideal of a semiring is a prime k-ideal.*

Proof. We can prove this affirmation along the lines of the proof of Lemma 2.6.5. However, in the present case, R is equal to the k-ideal generated by $S = \mathfrak{m} \cup \{a\}$ as a k-ideal. By Corollary 2.5.4, this means that $\sum e_k c_k + 1 = \sum f_l d_l$ for some $c_k, d_l \in S$ and $e_k, f_l \in R$. Multiplying with b yields $\sum be_k c_k + b = \sum bf_l d_l$. As reasoned in the proof of Lemma 2.6.5, $be_k c_k$ and $bf_l d_l$ are in $\mathfrak{m}$, and since $\mathfrak{m}$ is a k-ideal, $b \in \mathfrak{m}$ as desired. □

Lemma 2.6.7. *Let $f : R \to R'$ be a morphism of semirings and I an ideal of R'. Then $f^{-1}(I)$ is an ideal of R. If I is prime, then $f^{-1}(I)$ is prime. If I is a k-ideal, then $f^{-1}(I)$ is a k-ideal.*

Proof. We verify that $f^{-1}(I)$ is an ideal. Obviously, it contains 0. If $a,b \in f^{-1}(I)$ and $c \in R$, then $f(a+b) = f(a) + f(b) \in I$ and $f(ca) = f(c)f(a) \in I$. Thus $a+b, ca \in f^{-1}(I)$. This shows that $f^{-1}(I)$ is an ideal.

Assume that I is prime, i.e. $S = R' - I$ is a multiplicative set. Then $f^{-1}(S) = R - f^{-1}(I)$ is a multiplicative set of R and thus $f^{-1}(I)$ is a prime ideal of R.

Assume that I is a k-ideal and consider an equality $a + c = b$ in R with $a,b \in f^{-1}(I)$. Then $f(a) + f(c) = f(b)$ and $f(a), f(b) \in I$, which implies that $f(c) \in I$. Thus $c \in f^{-1}(I)$, which shows that $f^{-1}(I)$ is a k-ideal. □

Remark 2.6.8. There is also a concept of prime congruences. More precisely, there are two possible variants. Let $\mathfrak{c}$ be a congruence on R. Then $\mathfrak{c}$ is a *weak prime congruence on* R if $R/\mathfrak{c}$ is nontrivial and without zero divisors, and $\mathfrak{c}$ is a *strong prime congruence on* R if $R/\mathfrak{c}$ is integral.

However, we do not intend to discuss congruence schemes in these notes and therefore do not pursue the topic of prime congruences. Note that as of today, there is no satisfying theory of congruences schemes for semirings, but that such a theory relies on solving some open problems concerning the structure sheaf of congruence spectra. To explain this issue in more fancy words: one is led to work with a Grothendieck pre-topology on the category of semirings that is not subcanonical. This requires a sophisticated setup that establishes substitutes of certain standard facts for subcanonical topologies.

Exercise 2.6.9. Determine all prime (k-)ideals, all maximal (k-)ideals and all weak and strong prime congruences of $\mathbb{N}$ and $\mathbb{B}[A]$ where $A = \{1, \epsilon\}$ is the semigroup with $\epsilon^2 = \epsilon$. Let $f : \mathbb{N} \to \mathbb{Z}$ be the inclusion of the natural numbers into the integers. Describe the map $\mathfrak{p} \to f^{-1}(\mathfrak{p})$ from the set of prime ideals of $\mathbb{Z}$ to the set of prime ideals of $\mathbb{N}$ explicitly. Is it injective? Is it surjective?

Exercise 2.6.10. Let R be a semiring and I a proper (k-)ideal of R. Show that R has a maximal (k-)ideal that contains I. *Hint:* The usual proof for rings works also for this case. In particular, the claim relies on the axiom of choice aka Zorn's lemma.

Exercise 2.6.11. Let R be a semiring and I, J be ideals of R. We define their product $I \cdot J$ as the ideal generated by $\{ab | a \in I, b \in J\}$. Show that an ideal $\mathfrak{p}$ of R is prime if and only if $I \cdot J \subset \mathfrak{p}$ implies $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ for all ideals I and J of R.

2.7 Localizations

Definition 2.7.1. Let R be a semiring and $S \subset R$ be a *multiplicative subset of* R, i.e. a subset that contains 1 and is closed under multiplication. The *localization of* R *at* S is the quotient $S^{-1}R$ of $S \times R$ by the equivalence relation that identifies (s, r) with (s', r') whenever there is a $t \in S$ such that $tsr' = ts'r$ in R. We write $\frac{r}{s}$ for the equivalence class of (s, r). The addition and multiplication of $S^{-1}R$ are defined by the formulas

$$\frac{r}{s} + \frac{r'}{s'} = \frac{sr' + s'r}{ss'} \quad \text{and} \quad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{r'r}{ss'}.$$

The zero of $S^{-1}R$ is $\frac{0}{1}$ and its one is $\frac{1}{1}$.

We write $R[h^{-1}]$ for $S^{-1}R$ if $S = \{h^i\}_{i \in \mathbb{N}}$ for an element $h \in R$ and call $R[h^{-1}]$ the *localization of* R *at* h. We write $R_\mathfrak{p}$ for $S^{-1}R$ if $S = R - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of R and call $R_\mathfrak{p}$ the *localization of* R *at* $\mathfrak{p}$. Assume that $S = R - \{0\}$ is a multiplicative subset of R. Then we write $\mathrm{Frac}(R)$ for $S^{-1}R$ and call it the *semifield of fractions of* R.

If I is an ideal of R, then we write $S^{-1}I$ for the ideal of $S^{-1}R$ that is generated by $\{\frac{a}{1} | a \in I\}$.

Lemma 2.7.2. *Let R be a semiring, I an ideal of R and S a multiplicative subset of R. Then*

$$S^{-1}I = \left\{ \tfrac{a}{s} \in S^{-1}R \,\middle|\, a \in I, s \in S \right\}.$$

Proof. It is clear that $S^{-1}I$ contains the set $\{\frac{a}{1} | a \in I\}$ of generators of $S^{-1}I$. If we have proven that the set $I_S = \{\frac{a}{s} | a \in I, s \in S\}$ is an ideal, then it follows that it contains $S^{-1}I$. The reverse inclusion follows from the observation that for $\frac{a}{s} \in I_S$, we have $\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \in S^{-1}I$.

We are left with showing that I_S is an ideal. It obviously contains $\frac{0}{1}$. Given $\frac{a}{s} \in I_S$ and $\frac{b}{t} \in S^{-1}R$, then $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in I_S$ since $ab \in I$. Given $\frac{a}{s}, \frac{b}{t} \in I_S$, then $a, b \in I$ and $ta + sb \in I$. Thus $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{st}$ is an element of I_S. This verifies that I_S is an ideal of $S^{-1}I$ and finishes the proof of the lemma. □

Exercise 2.7.3. Let R be a semiring and S a multiplicative subset of R. Show that the map $\iota_S : R \to S^{-1}R$, defined by $\iota_S(a) = \frac{a}{1}$, is a morphism of semirings that maps S to the units of $S^{-1}R$. Show that it satisfies the usual universal property of localizations: every morphism $f : R \to R'$ of semirings that maps S to the units of R' factors uniquely through ι_S. Show that ι_S is an epimorphism.

Exercise 2.7.4. The subset $S = R - \{0\}$ is a multiplicative subset if and only if R is nontrivial and without zero divisors. Assuming that S is a multiplicative subset, show that $\operatorname{Frac} R$ is a semifield. Show that the morphism $\iota_S : R \to \operatorname{Frac}(R)$ is injective if and only if R is integral. Describe an example where $R \to \operatorname{Frac}(R)$ is not injective.

Proposition 2.7.5. *Let R be a semiring, S a multiplicative subset of R and $\iota_S : R \to S^{-1}R$ the localization morphism. Then the maps*

$$\begin{array}{ccc} \{\textit{prime ideals } \mathfrak{p} \textit{ of } R \textit{ with } \mathfrak{p} \cap S = \emptyset\} & \longleftrightarrow & \{\textit{prime ideals of } S^{-1}R\} \\ \mathfrak{p} & \overset{\Phi}{\longmapsto} & S^{-1}\mathfrak{p} \\ \iota_S^{-1}(\mathfrak{q}) & \overset{\Psi}{\longleftarrow\!\shortmid} & \mathfrak{q} \end{array}$$

are mutually inverse bijections. A prime ideal $\mathfrak{p}$ of R with $\mathfrak{p} \cap S = \emptyset$ is a k-ideal if and only if $S^{-1}\mathfrak{p}$ is a k-ideal.

Proof. To begin with, we verify that both Φ and Ψ are well-defined. Let $\mathfrak{p}$ be a prime ideal of R such that $\mathfrak{p} \cap S = \emptyset$. Then $S^{-1}\mathfrak{p} = \{\frac{a}{s} | a \in \mathfrak{p}, s \in S\}$ by Lemma 2.7.2. Consider $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ such that $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in S^{-1}\mathfrak{p}$, i.e. $ab \in \mathfrak{p}$. Then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ and thus $\frac{a}{s} \in S^{-1}\mathfrak{p}$ or $\frac{b}{t} \in S^{-1}\mathfrak{p}$. This shows that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}R$ and that Φ is well-defined.

Let $\mathfrak{q}$ be a prime ideal of $S^{-1}R$. By Lemma 2.6.7, $\iota_S^{-1}(\mathfrak{q})$ is a prime ideal of R. Note that $\mathfrak{q}$ is proper and does not contain any element of the form $\frac{s}{t}$ with $s, t \in S$ since $\frac{t}{s} \cdot \frac{s}{t} = 1$. Thus $\iota_S^{-1}(\mathfrak{q})$ intersects S trivially. This shows that Ψ is well-defined.

We continue with the proof that $\Psi \circ \Phi$ is the identity, i.e. $\iota_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R that does not intersect S. The inclusion $\mathfrak{p} \subset \iota_S^{-1}(S^{-1}\mathfrak{p})$ is trivial. The reverse inclusion can be shown as follows. The set $\iota_S^{-1}(S^{-1}\mathfrak{p})$ consists of all elements $a \in R$ such that $\frac{a}{1} = \frac{b}{s}$ for some $b \in \mathfrak{p}$ and $s \in S$. This equation says that there is a $t \in S$ such that $tsa = tb$. Since $b \in \mathfrak{p}$, we have $tsa = tb \in \mathfrak{p}$. Since $ts \notin \mathfrak{p}$, we have $a \in \mathfrak{p}$, as desired.

We continue with the proof that $\Phi \circ \Psi$ is the identity, i.e. $S^{-1}\iota_S^{-1}(\mathfrak{q}) = \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of $S^{-1}R$. The inclusion $S^{-1}\iota_S^{-1}(\mathfrak{q}) \subset \mathfrak{q}$ is trivial. The reverse inclusion can be shown as follows. Let $\frac{a}{s} \in \mathfrak{q}$. Then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in \mathfrak{q}$ and $a \in \iota_S^{-1}\mathfrak{q}$. Thus $\frac{a}{s} \in S^{-1}\iota_S^{-1}(\mathfrak{q})$, as desired. This concludes the proof of the first claim of the proposition.

We continue with the proof that a prime ideal $\mathfrak{p}$ of R with $\mathfrak{p} \cap S = \emptyset$ is a k-ideal if and only if $S^{-1}\mathfrak{p}$ is a k-ideal. First assume that $S^{-1}\mathfrak{p}$ is a k-ideal and consider an equality $a + c = b$ with $a, b \in \mathfrak{p}$. Then we have $\frac{a}{1} + \frac{c}{1} = \frac{b}{1}$ with $\frac{a}{1}, \frac{b}{1} \in S^{-1}\mathfrak{p}$. Since $S^{-1}\mathfrak{p}$ is a k-ideal, we have $\frac{c}{1} \in S^{-1}\mathfrak{p}$ and thus $c \in \mathfrak{p}$. This shows that $\mathfrak{p}$ is a k-ideal.

Conversely, assume that $\mathfrak{p}$ is a k-ideal and consider an equality $\frac{a}{s} + \frac{c}{u} = \frac{b}{t}$ with $\frac{a}{s}, \frac{b}{t} \in S^{-1}\mathfrak{p}$. This means that $wtua + wstc = wsub$ for some $w \in S$. Since $wtua$ and $wsub$ are elements of the k-ideal $\mathfrak{p}$, also $wstc \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime and $wst \notin \mathfrak{p}$, we have $c \in \mathfrak{p}$ and thus $\frac{c}{u} \in S^{-1}\mathfrak{p}$, as desired. This finishes the proof of the proposition. □

Residue fields

Let R be a semiring, $\mathfrak{p}$ a prime ideal of R and $S = R - \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the complement of the units of $S^{-1}R$ and therefore its unique maximal ideal.

Definition 2.7.6. Let R be a semiring and $\mathfrak{p}$ a prime ideal of R. The *residue field at* $\mathfrak{p}$ is the semiring $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{c}(S^{-1}\mathfrak{p})$ where S is the complement of $\mathfrak{p}$ in R and $\mathfrak{c}(S^{-1}\mathfrak{p})$ is the congruence on $R_{\mathfrak{p}}$ that is generated by $S^{-1}\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of a semiring R. Then the residue field at $\mathfrak{p}$ comes with a canonical morphism $R \to k(\mathfrak{p})$, which is the composition of the localization map $R \to R_{\mathfrak{p}}$ with the quotient map $R_{\mathfrak{p}} \to k(\mathfrak{p})$. Note that the residue field $k(\mathfrak{p})$ can be the trivial semiring in case that $\mathfrak{p}$ is not a k-ideal. More precisely, we have the following.

Corollary 2.7.7. *Let R be a semiring, $\mathfrak{p}$ a prime ideal of R and $S = R - \mathfrak{p}$. Then the residue field $k(\mathfrak{p})$ is a semifield if $\mathfrak{p}$ is a k-ideal and trivial if not.*

Proof. First assume that $\mathfrak{p}$ is a prime k-ideal. Then $\mathfrak{p}$ is the maximal prime ideal that does not intersect S and thus $\mathfrak{m} = S^{-1}\mathfrak{p}$ is the unique maximal of $S^{-1}R$. By Proposition 2.7.5, $\mathfrak{m}$ is a k-ideal. Thus the kernel of $S^{-1}R \to k(\mathfrak{p})$ is $\mathfrak{m}$, which shows that $k(\mathfrak{p})$ is not trivial. Since $(S^{-1}R)^{\times} = S^{-1}R - \mathfrak{m}$, we see that $(S^{-1}R)^{\times} \to k(\mathfrak{p}) - \{0\}$ is surjective, which shows that all nonzero elements of $k(\mathfrak{p})$ are invertible, i.e. $k(\mathfrak{p})$ is a semifield.

Next assume that $\mathfrak{p}$ is not a k-ideal. By Proposition 2.7.5, $\mathfrak{m} = S^{-1}\mathfrak{p}$ is not a k-ideal, which means that the kernel of $S^{-1}R \to k(\mathfrak{p})$ is strictly larger than $\mathfrak{m}$ and therefore contains a unit of $S^{-1}R$. This shows that $k(x)$ must be trivial. □

Corollary 2.7.8. *Let R be a nontrivial semiring. Then there exists a morphism $R \to k$ to a semifield k.*

Proof. By Exercise 2.6.10, R has a maximal k-ideal $\mathfrak{p}$. By Lemma 2.6.6, $\mathfrak{p}$ is prime. By Corollary 2.7.7, $k(\mathfrak{p})$ is a semifield, and the canonical morphism $R \to k(\mathfrak{p})$ verifies the claim of the corollary. □

Exercise 2.7.9. Let R be a semiring and $\mathfrak{p}$ a prime k-ideal of R. Show that $R/\mathfrak{p}$ is nontrivial and without zero divisors and that $k(\mathfrak{p})$ is isomorphic to $\mathrm{Frac}(R/\mathfrak{p})$. What happens if $\mathfrak{p}$ is a prime ideal that is not a k-ideal?

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Chapter 3

Monoids with zero

In this chapter, we introduce and investigate monoids with zero. As we will see that monoids with zero behave like semirings in many aspects. In particular, most results of Chapter 2 have an analogue for monoids with zero. We review these facts in the following and emphasize the analogy with semirings by a similar formal structure of this chapter with Chapter 2. We will see, though, that several facts and constructions are much simpler for monoids than for semirings.

3.1 The category of monoids with zero

Definition 3.1.1. A *monoid with zero* is a set A together with an associative and commutative multiplication $\cdot : A \times A \to A$ and two constants 0 and 1 such that $0 \cdot a = 0$ and $1 \cdot a = a$ for all $a \in A$. We often write ab for $a \cdot b$.

A *morphism between monoids with zero* A_1 *and* A_2 is a map $f : A_1 \to A_2$ such that $f(0) = 0$, $f(1) = 1$ and $f(ab) = f(a)f(b)$. This defines the category Mon of monoids with zero.

Let A be a monoid with zero. A *submonoid of* A is a multiplicatively closed subset that contains 0 and 1. The *unit group of* A is the subset $A^\times$ of invertible elements of A.

Note that the multiplication of A restricts to $A^\times$ and turns it into an abelian group. Note further that the constants 0 and 1 of a monoid with zero A are uniquely determined by the properties $0 \cdot a = 0$ and $1 \cdot a = a$. Sometimes, we take the liberty to omit an explicit description of these constants and we call a monoid with zero simply a monoid if it clearly contains a zero. Note, however, that the property $f(0) = 0$ of a morphism of monoids with zero is not automatically implied by the other axioms; in other words, not every monoid morphism between monoids with zero is a morphism of monoids with zero.

Example 3.1.2. Every semiring R is a monoid with zero if we omit the addition from the structure. We write $R^\bullet$ for the multiplicative monoid of R.

Given a (multiplicatively written) abelian semigroup A with unit 1, we obtain a monoid with zero $A_0 = A \cup \{0\}$ by adding an element 0 satisfying $0 \cdot a = 0$ for all $a \in A_0$.

The *trivial monoid with zero* $\{0 = 1\}$ is a terminal object in Mon. The so-called *field with one element* $\mathbb{F}_1 = \{0, 1\}$ is initial in Mon.

Exercise 3.1.3. Show that Mon is complete and cocomplete. The proof can be done in analogy to the case of semirings, cf. Exercise 2.1.6. In particular, the product of monoids A_i is represented by the Cartesian product $\prod A_i$ and their coproduct is a union over finite tensor products over $\mathbb{F}_1$; the equalizer of two morphisms $f, g : A \to B$ is represented by $\mathrm{eq}(f, g) = \{a \in A | f(a) = g(a)\}$ and their coequalizer is the quotient of B by the congruence generated by the relations $f(a) \sim g(a)$ for $a \in A$.

Definition 3.1.4. A monoid with zero A is *without zero divisors* if for any $a,b \in R$, the equality $ab = 0$ implies that $a = 0$ or $b = 0$. It is *integral* (or *multiplicatively cancellative*) if $0 \neq 1$ and for any $a,b,c \in R$ the equality $ac = bc$ implies $c = 0$ or $a = b$.

Lemma 3.1.5. *An integral monoid with zero is without zero divisors.*

Proof. If R is integral and $ab = 0$, then $ab = 0 = 0 \cdot b$ implies $b = 0$ or $a = 0$. □

Note that as in the case of semirings, a nontrivial monoid with zero and without zero divisor is in general not integral. An example of such a monoid is a semiring with the corresponding properties, e.g. the multiplicative monoid $\mathbb{T}[T]^\bullet$ of the tropical polynomial algebra $\mathbb{T}[T]$.

3.2 Tensor products and free monoids with zero

Definition 3.2.1. Let $f_A : C \to A$ and $f_B : C \to B$ be two morphisms of monoids with zero. The *tensor product of A and B over C* is the set

$$A \otimes_C B \;=\; A \times B / \sim$$

where the equivalence relation $\sim$ is generated by relations of the form $(f_A(c)a, b) \sim (a, f_B(c)b)$ where $a \in A$, $b \in B$ and $c \in C$. We denote the equivalence class of (a,b) by $a \otimes b$. The multiplication of $A \otimes_C B$ is defined by the formula

$$(a \otimes b) \cdot (a' \otimes b') \;=\; aa' \otimes bb'.$$

Its zero is $0 \otimes 0$ and its one is $1 \otimes 1$. The tensor product $A \otimes_C B$ comes with the canonical maps $\iota_A : A \to A \otimes_C B$, sending a to $a \otimes 1$, and $\iota_B : B \to A \otimes_C B$, sending b to $1 \otimes b$.

Exercise 3.2.2. Verify that $A \otimes_C B$ is indeed a monoid with zero and that the canonical maps ι_A and ι_B are morphisms.

Formulate and prove the usual properties of the tensor product: (1) The tensor product is the colimit (or pushout) of the diagram $A \xleftarrow{f_A} C \xrightarrow{f_B} B$; (2) every C-bilinear morphism from $A \times B$ defines a unique morphism from $A \otimes_C B$.

Exercise 3.2.3. Let B be monoids with zero and A be a (multiplicatively written) abelian semigroup with neutral element 1. Let $A_0 = A \cup \{0\}$ be the associated monoid with zero; cf. Example 3.1.2. Let $\mathbb{F}_1 \to A_0$ and $\mathbb{F}_1 \to B$ the unique morphisms from the initial object $\mathbb{F}_1$ into A_0 and B, respectively.

Show that the underlying set of $B \otimes_{\mathbb{F}_1} A_0$ is the *smash product* $B \wedge A_0$, which is the quotient of $B \times A_0$ by the equivalence relation generated by $(0,a) \sim (b,0)$ for all $a \in A_0$ and $b \in B$.

Let $B[A] = B \otimes_{\mathbb{F}_1} A_0$, let $\iota_B : B \to B[A]$ be the canonical map and let $\bar{\iota}_A : A \to B[A]$ be the composition of the inclusion $A \to A_0$ followed by the canonical map $A_0 \to B[A]$. Conclude from Exercise 3.2.2 that $B[A] = B \otimes_{\mathbb{F}_1} A_0$ satisfies the following universal property: for every morphism $f_B : B \to C$ of monoids with zero and every multiplicative map $f_A : A \to C$ with $f_A(1) = 1$, there is a unique morphism $F : B[A] \to C$ of monoids with zero such that $f_B = F \circ \iota_B$ and $f_A = F \circ \bar{\iota}_A$. Conclude that $B[A]$ is the analogue of a semigroup algebra for monoids with zero; cf. section 2.3.

Definition 3.2.4. Given a monoid with zero A and a set $\{T_i\}_{i\in I}$, the *free monoid with zero over A in $\{T_i\}$* is the monoid with zero $A[T_i]_{i\in I} = A\otimes_{\mathbb{F}_1} S_0$ where $S = \{\prod T_i^{e_i}\}_{(e_i)\in\bigoplus\mathbb{N}}$ is the multiplicative semigroup of all monomials $\prod T_i^{e_i}$ in the T_i.

If $I = \{1,\dots,n\}$, then we write $A[T_1,\dots,T_n]$ for $A[T_i]_{i\in I}$. We write $a\prod T_i^{e_i}$ for $a\otimes\prod T_i^{e_i}$ and a for the element $a\prod T_i^0$, which we call it a *constant monomial of* $A[T_i]_{i\in I}$. We write $aT_{i_1}^{e_{i_1}}\cdots T_{i_n}^{e_{i_n}}$ for $a\prod T_i^{f_i}$ with $f_{i_k} = e_{i_k}$ for $k = 1,\dots,n$ and $f_j = 0$ otherwise.

Exercise 3.2.5. Let A be a monoid with zero and $\{T_i\}_{i\in I}$ a set. Let $\iota_A : A\to A[T_i]$ be the canonical morphism of monoids with zero and $\iota_0 : \{T_i\}\to A[T_i]$ the canonical inclusion. Show that for every morphism $f_A : A\to B$ of monoids with zero and every map $f_0 : \{T_i\}\to B$, there is a unique morphism $f : A[T_i]\to B$ of monoids with zero such that $f_A = f\circ\iota_A$ and $f_0 = f\circ\iota_0$.

Exercise 3.2.6. Let $f : R_1\to R_2$ be a morphism of semirings. Show that f is also a morphism of the underlying monoids, which we denote by $f^\bullet : R_1^\bullet\to R_2^\bullet$. Show that this defines a functor $(-)^\bullet : \text{SRings}\to\text{Mon}$.

This functor has left adjoint, which can be described as follows. Given a monoid A with zero 0_A, we define A^+ as the semiring $\mathbb{N}[A]/\mathfrak{c}(0_A)$, i.e. the semigroup algebra of A over $\mathbb{N}$ whose zero we identify with 0_A. Show that a morphism $f : A_1\to A_2$ of monoids with zero defines a semiring morphism $f^+ : A_1^+\to A_2^+$ by linear extension. Show that this defines a functor $(-)^+ : \text{Mon}\to\text{SRings}$, which is left adjoint to $(-)^\bullet : \text{SRings}\to\text{Mon}$, i.e. are bijections

$$\mathrm{Hom}_{\mathrm{Mon}}(A,R^\bullet) \;\xrightarrow{\sim}\; \mathrm{Hom}_{\mathrm{SRings}}(A^+,R)$$

for all monoids with zero A and every semirings R, which are functorial in A and R.

Exercise 3.2.7. Show that the multiplicative monoid $\mathbb{N}^\bullet$ of $\mathbb{N}$ is isomorphic to $\mathbb{F}_1[T_p]_{p\in\mathcal{P}}$ where $\mathcal{P}$ is the set of prime numbers in $\mathbb{N}$.

3.3 Congruences of monoids

Definition 3.3.1. Let A be a monoid with zero. A *congruence on A* is an equivalence relation $\mathfrak{c}$ on A that is *multiplicative*, i.e. (a,b) and (c,d) in $\mathfrak{c}$ imply $(ac,bd)\in\mathfrak{c}$ for all $a,b,c,d\in A$.

Example 3.3.2. Let R be a semiring and $\mathfrak{c}$ a congruence on R. Then $\mathfrak{c}$ is also a congruence on the monoid $R^\bullet$.

Exercise 3.3.3. Let $k,n\in\mathbb{N}$. Show that the sets

$$\mathfrak{c}_n \;=\; \big\{\,(a,b)\in\mathbb{F}_1[T]\times\mathbb{F}_1[T]\,\big|\,a=b \text{ or } a,b\in\{T^k|k\geqslant n\}\cup\{0\}\,\big\}$$

and

$$\mathfrak{c}_{k,n} \;=\; \big\{\,(T^{m+rk},T^{m+sk})\in\mathbb{F}_1[T]\times\mathbb{F}_1[T]\,\big|\,m,r,s\in\mathbb{N},\text{ and } m\geqslant n \text{ or } r=s=0\,\big\}\;\cup\;\big\{\,(0,0)\,\big\}$$

are congruences on the free monoid with zero $\mathbb{F}_1[T]$ in T over $\mathbb{F}_1$ for all $k,n\geqslant 0$. Show that every congruence of $\mathbb{F}_1[T]$ is of this form.

Let $\mathfrak{c}$ be a congruence on A. Similar to the case of congruences for semirings, we write $a\sim_{\mathfrak{c}} b$, or simply $a\sim b$, to express that (a,b) is an element of $\mathfrak{c}$. The following proposition shows that congruences define quotients of monoids with zero.

Proposition 3.3.4. *Let A be a monoid with zero and $\mathfrak{c}$ be a congruence on A. Then the association $[a]\cdot[b]=[ab]$ is well-defined on equivalence classes of $\mathfrak{c}$ and turn the quotient $A/\mathfrak{c}$ into a monoid with zero $[0]$ and neutral element $[1]$.*

The quotient map $\pi : A \to A/\mathfrak{c}$ is a morphism of monoids with zero that satisfies the following universal property: every morphism $f : A \to B$ such that $f(a) = f(b)$ whenever $a \sim_{\mathfrak{c}} b$ factors uniquely through π.

Proof. Given $a \sim a'$ and $b \sim b'$, we have $ab \sim a'b'$. Thus the multiplication of $A/\mathfrak{c}$ does not depend on the choice of representative and is therefore well-defined. It follows immediately that A is a monoid with zero $[0]$ and neutral element $[1]$ and that π a morphism of monoids with zero.

Let $f : A \to B$ be a morphism such that $f(a) = f(b)$ whenever $a \sim b$ in $\mathfrak{c}$. For f to factor into $\overline{f}\circ\pi$ for a morphism $\overline{f} : A/\mathfrak{c} \to B$, it is necessary that $\overline{f}([a]) = \overline{f}\circ\pi(a) = f(a)$. This shows that $\overline{f}$ is unique if it exists. Since $a \sim b$ implies $f(a) = f(b)$, we conclude that $\overline{f}$ is well-defined as a map. The verification of the axioms of a morphism are left as an exercise. □

Example 3.3.5. Let A be a monoid with zero and without zero divisors. Then $\mathfrak{c} = \{(a,b) \in A\times A | a \neq 0 \neq b\} \cup \{(0,0)\}$ is a congruence. The quotient $A/\mathfrak{c}$ is isomorphic to $\mathbb{F}_1$.

Exercise 3.3.6. Describe the quotients $\mathbb{F}_1[T]/\mathfrak{c}_{k,n}$ for $k,n \in \mathbb{N}$ where $\mathfrak{c}_{k,n}$ are the congruences from Exercise 3.3.3.

Definition 3.3.7. Let $f : A \to B$ be a morphism of monoids with zero. The *congruence kernel of* f is the relation $\mathfrak{c}(f) = \{(a,b) \in A\times A | f(a) = f(b)\}$ on A.

Lemma 3.3.8. *The congruence kernel $\mathfrak{c}(f)$ of a morphism $f : A \to B$ of monoids with zero is a congruence on A.*

Proof. That $\mathfrak{c} = \mathfrak{c}(f)$ is an equivalence relation follows from the following calculations: $f(a) = f(a)$ (reflexive); $f(a) = f(b)$ implies $f(b) = f(a)$ (symmetry); $f(a) = f(b)$ and $f(b) = f(c)$ imply $f(a) = f(c)$ (transitive). Multiplicativity follows from: $f(a) = f(b)$ and $f(c) = f(d)$ imply $f(ac) = f(a)\cdot f(c) = f(b)\cdot f(d) = f(bd)$. This shows that $\mathfrak{c}$ is a congruence. □

As a consequence of this lemma, we see that for a monoid with zero A, the associations

$$\begin{array}{ccc} \{\text{congruences on } A\} & \longleftrightarrow & \{\text{quotients of } A\} \\ \mathfrak{c} & \longmapsto & A \to A/\mathfrak{c} \\ \mathfrak{c}(\pi) & \longleftarrow\!\shortmid & \pi : A \twoheadrightarrow B \end{array}$$

are mutually inverse bijections. We will see in section 3.4 that we have a similar discrepancy between quotients and ideals as in the case of semirings. In so far, one has to work with congruences when one wants to describe quotients of monoids with zero.

Lemma 3.3.9. *Let A be a monoid with zero and $S \subset A\times A$ a subset. Then there is a smallest congruence $\mathfrak{c} = \langle S\rangle$ containing S. The quotient map $\pi : A \to A/\langle S\rangle$ satisfies the following universal property: every morphisms $f : A \to B$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$ factors uniquely through π.*

Proof. It is readily verified that the intersection of congruences is again a congruence. As a consequence, the intersection of all congruences containing S is the smallest congruence containing S.

Given any morphism $f : A \to B$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$, then the congruence kernel $\mathfrak{c}(f)$ must contain S and thus $\mathfrak{c} = \langle S\rangle$. Using Proposition 2.4.4, we see that f factors uniquely through π. □

This lemma shows that we can construct new monoids with zero from known ones by prescribing a number of relations: let A be a monoid with zero and $\{a_i \sim b_i\}$ a set of relations on A, i.e. $S = \{(a_i,b_i)\}$ is a subset of $A \times A$. Then we define $A/\langle a_i \sim b_i \rangle$ as the quotient monoid $A/\langle S \rangle$.

Example 3.3.10. In $A = \mathbb{F}_1[T]/\langle T^2 \sim T\rangle$, we have $[T^{2+i}] = [T^{1+i}]$ for all $i \geqslant 0$, thus A consists of the residue classes $[0]$, $[1]$ and $[T]$, and $[T]^2 = [T]$ is an idempotent element of A.

Exercise 3.3.11. Let A be a monoid with zero and $\mathfrak{c}$ a congruence on A. Let $\mathfrak{c}^+$ be the congruence on the semiring A^+ that is generated by $\mathfrak{c} \subset A \times A \subset A^+ \times A^+$. Show that $A^+/\mathfrak{c}^+$ is isomorphic to $(A/\mathfrak{c})^+$.

3.4 Ideals

Definition 3.4.1. Let A be a monoid with zero. An *ideal of A* is a subset I of A such that 0 and ab are elements of I for all $a \in I$ and $b \in A$. Let $f : A \to B$ be a morphism of monoids with zero. The *(ideal) kernel of f* is the inverse image $\ker(f) = f^{-1}(0)$ of 0.

Let S be a subset of A. The *ideal generated by S* is the set $\langle S \rangle = \{as \in A | a \in A, s \in S \cup \{0\}\}$. The *congruence generated by S* is the congruence $\mathfrak{c}(S)$ generated by $\{(a,0)|a \in S\}$.

Note that $\langle S \rangle$ is the smallest ideal of A containing S. In particular, we have $\langle \emptyset \rangle = \{0\}$. Note further that the congruence generated by S is the set

$$\mathfrak{c}(S) \;=\; \big\{ (a,b) \,\big|\, a,b \in \langle S \rangle \big\} \;\cup\; \big\{ (a,a) \,\big|\, a \in A \big\}.$$

Exercise 3.4.2. Describe all ideals of $\mathbb{F}_1[T]$ and of $\mathbb{N}^\bullet$. Determine which congruences on $\mathbb{F}_1[T]$ are geneerated by ideals, cf. Exercise 3.3.3.

Proposition 3.4.3. *The kernel* $\ker(f)$ *of a morphism of $f : A \to B$ is an ideal and every ideal appears as a kernel. More precisely, if I is an ideal of A and $\mathfrak{c} = \mathfrak{c}(I)$ is the congruence generated by I, then I is the kernel of $\pi : A \to A/\mathfrak{c}$ and $\pi(a) = \pi(b)$ if and only if $a,b \in I$ or $a = b$.*

Proof. We begin with the verification that $\ker(f)$ is an ideal. Clearly $0 \in \ker(f)$. Let $a \in \ker(f)$ and $b \in R$. Then $f(ab) = f(a)f(b) = 0 \cdot f(b) = 0$ and $ab \in \ker(f)$. Thus $\ker(f)$ is an ideal.

It is easily verified that $\mathfrak{c} = \mathfrak{c}(I)$ has the explicit description

$$\mathfrak{c} \;=\; \big\{ (a,b) \in A \times A \,|\, a,b \in I \text{ or } a = b \big\}.$$

It follows that $\pi(a) = \pi(b)$ if and only if $a,b \in I$ or $a = b$, and that $\ker f = \{a \in A | \pi(a) = 0\} = I$, as claimed. $\square$

Remark 3.4.4. As a consequence of Proposition 3.4.3, we see that the quotient $A/\mathfrak{c}(I)$ of A by an ideal I contracts all elements of the ideal I, but does not identify any other elements. In other words, $A/\mathfrak{c}(I)$ stays in bijection with $\{0\} \cup (A - I)$.

We summarize: with a congruence $\mathfrak{c}$ on A, we can associate the kernel of the projection $\pi_\mathfrak{c} : A \to A/\mathfrak{c}$, which is an ideal; with an ideal I, we can associate the congruence $\mathfrak{c}(I)$ generated by I. We have that the kernel of $A \to A/\mathfrak{c}(I)$ is I and the congruence $\mathfrak{c}(\ker \pi_\mathfrak{c})$ is contained in $\mathfrak{c}$, but in general not equal to $\mathfrak{c}$. This leads to the following picture.

$$\begin{array}{ccc} \textbf{“kernels”} & & \textbf{“quotients”} \\ \big\{ \text{ideals of } A \big\} & \overset{\twoheadleftarrow}{\hookrightarrow} & \big\{ \text{congruences on } A \big\} \end{array}$$

Exercise 3.4.5. Compare the ideals of $\mathbb{F}_1[T]$ with the congruences on $\mathbb{F}_1[T]$; cf. Exercises 3.3.3 and 3.4.2. Do the same exercise for $\mathbb{F}_1[T_1,T_2]$.

3.5 Prime ideals

Definition 3.5.1. Let A be a monoid with zero. An ideal I of A is *proper* if it is not equal to A. It is *maximal* if it is proper and if $I \subset J$ implies $I = J$ for any other proper ideal of A. It is *prime* if its complement $S = A - I$ is a multiplicative subset of R.

Let A be a monoid with zero. Then $\mathfrak{m} = A - A^\times$ is an ideal of A, which is necessarily the unique maximal ideal of A. This shows that every monoid with zero A is *local*, i.e. A contains a unique maximal ideal $\mathfrak{m}$ and it satisfies $A = A^\times \cup \mathfrak{m}$.

Exercise 3.5.2. Show that for every subset $J \subset \{1,\dots,n\}$, the ideals

$$\langle T_i | i \in J \rangle \;=\; \{0\} \cup \Big\{ \prod_{i=1}^n T_i^{e_i} \in \mathbb{F}_1[T_1,\dots,T_n] \,\Big|\, e_i > 0 \text{ for some } i \in J \Big\}$$

of $\mathbb{F}_1[T_1,\dots,T_n]$ are prime ideals and that every prime ideal of $\mathbb{F}_1[T_1,\dots,T_n]$ is of this form.

Lemma 3.5.3. *Let A be a monoid with zero and I an ideal of A. Then I is prime if and only if A/I is nontrivial and without zero divisors, and I is maximal if and only if $A/I = (A/I)^\times \cup \{[0]\}$.*

Proof. The ideal I is prime if and only if for all $a,b \in A$, $ab \in I$ implies that $a \in I$ or $b \in I$. Passing to the quotient A/I, this means that $[ab] = [0]$ implies $[a] = [0]$ or $[b] = [0]$ where we use that the kernel of $A \to A/I$ is I, cf. Proposition 3.4.3. This latter condition is equivalent to A/I being nontrivial and without zero divisors.

As observed above, I is maximal if and only if $I = A - A^\times$. In this case, A/I is isomorphic to $(A^\times)_0 = A^\times \cup \{0\}$ and thus satisfies $A/I = (A/I)^\times \cup \{[0]\}$. Conversely, if $[a]\cdot[b] = 1$ in A/I, then $ab = 1$ in A since $[a] \neq [0] \neq [b]$ and thus $[a] = \{a\}$ and $[b] = \{b\}$ by Proposition 3.4.3. Thus if $A/I = (A/I)^\times \cup \{[0]\}$, then $I = \ker(A \to A/I) = A - A^\times$. □

Lemma 3.5.4. *Every maximal ideal is a prime ideal.*

Proof. This follows immediately from the characterization of the unique maximal ideal as the complement of the unit group and the fact that the product of non-units is a non-unit. □

Lemma 3.5.5. *Let $f : A \to B$ be a morphism of monoids with zero and I an ideal of B. Then $f^{-1}(I)$ is an ideal of A. If I is prime, then $f^{-1}(I)$ is prime.*

Proof. We verify that $f^{-1}(I)$ is an ideal. Obviously, it contains 0. If $a \in f^{-1}(I)$ and $b \in A$, then $f(ab) = f(a)f(b) \in I$ and $ab \in f^{-1}(I)$. This shows that $f^{-1}(I)$ is an ideal.

Assume that I is prime, i.e. $S = B - I$ is a multiplicative set. Then $f^{-1}(S) = A - f^{-1}(I)$ is a multiplicative set of A and thus $f^{-1}(I)$ is a prime ideal of A. □

Remark 3.5.6. Similar to the case of semirings, there exist two concepts of prime congruences for monoids with zero. Namely, a congruence $\mathfrak{c}$ on a monoid with zero A is a *weak prime congruence on* A if $A/\mathfrak{c}$ is nontrivial and without zero divisors, and $\mathfrak{c}$ is a *strong prime congruence on* A if $A/\mathfrak{c}$ is integral.

3.6 Localizations

Definition 3.6.1. Let A be a monoid with zero and $S \subset A$ be a *multiplicative subset of* A, i.e. a subset that contains 1 and is closed under multiplication. The *localization of* A *at* S is the quotient $S^{-1}A$ of $S \times A$ by the equivalence relation that identifies (s,a) with (s',a') whenever there is a $t \in S$ such that $tsa' = ts'a$ in A. We write $\frac{a}{s}$ for the equivalence class of (s,a). The multiplication of $S^{-1}A$ is defined by the formula $\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$. The zero of $S^{-1}A$ is $\frac{0}{1}$ and its one is $\frac{1}{1}$.

We write $A[h^{-1}]$ for $S^{-1}A$ if $S = \{h^i\}_{i\in\mathbb{N}}$ for an element $h \in A$ and call $A[h^{-1}]$ the *localization of* A *at* h. We write $A_{\mathfrak{p}}$ for $S^{-1}A$ if $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of A and call $A_{\mathfrak{p}}$ the *localization of* A *at* $\mathfrak{p}$.

If I is an ideal of A, then we write $S^{-1}I$ for the ideal of $S^{-1}A$ that is generated by $\{\frac{a}{1} | a \in I\}$.

Lemma 3.6.2. *Let A be a monoid with zero, I an ideal of A and S a multiplicative subset of A. Then*

$$S^{-1}I = \left\{ \tfrac{a}{s} \in S^{-1}A \,\middle|\, a \in I, s \in S \right\}.$$

Proof. It is clear that $S^{-1}I$ contains the set $\{\frac{a}{1} | a \in I\}$ of generators of $S^{-1}I$. If we have proven that the set $I_S = \{\frac{a}{s} | a \in I, s \in S\}$ is an ideal, then it follows that it contains $S^{-1}I$. The reverse inclusion follows from the observation that for $\frac{a}{s} \in I_S$, we have $\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \in S^{-1}I$.

We are left with showing that I_S is an ideal. It obviously contains $\frac{0}{1}$. Given $\frac{a}{s} \in I_S$ and $\frac{b}{t} \in S^{-1}A$, then $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in I_S$ since $ab \in I$. This verifies that I_S is an ideal of $S^{-1}I$ and finishes the proof of the lemma. □

Exercise 3.6.3. Let A be a monoid with zero and S a multiplicative subset of A. Show that the map $\iota_S : A \to S^{-1}A$, defined by $\iota_S(a) = \frac{a}{1}$, is a morphism of monoids with zero that maps S to the units of $S^{-1}A$. Show that it satisfies the usual universal property of localizations: every morphism $f : A \to B$ of monoids with zero that maps S to the units of B factors uniquely through ι_S. Show that ι_S is an epimorphism.

Proposition 3.6.4. *Let A be a monoid, S a multiplicative subset of A and $\iota_S : A \to S^{-1}A$ the localization morphism. Then the maps*

$$\begin{array}{ccc} \{\textit{prime ideals } \mathfrak{p} \textit{ of } A \textit{ with } \mathfrak{p} \cap S = \emptyset\} & \longleftrightarrow & \{\textit{prime ideals of } S^{-1}A\} \\ \mathfrak{p} & \stackrel{\Phi}{\longmapsto} & S^{-1}\mathfrak{p} \\ \iota_S^{-1}(\mathfrak{q}) & \stackrel{\Psi}{\longleftarrow\!\shortmid} & \mathfrak{q} \end{array}$$

are mutually inverse bijections.

Proof. To begin with, we verify that both Φ and Ψ are well-defined. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Then $S^{-1}\mathfrak{p} = \{\frac{a}{s} | a \in \mathfrak{p}, s \in S\}$ by Lemma 3.6.2. Consider $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$ such that $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in S^{-1}\mathfrak{p}$, i.e. $ab \in \mathfrak{p}$. Then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ and thus $\frac{a}{s} \in S^{-1}\mathfrak{p}$ or $\frac{b}{t} \in S^{-1}\mathfrak{p}$. This shows that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$ and that Φ is well-defined.

Let $\mathfrak{q}$ be a prime ideal of $S^{-1}A$. By Lemma 3.5.5, $\iota_S^{-1}(\mathfrak{q})$ is a prime ideal of A. Note that $\mathfrak{q}$ is proper and does not contain any element of the form $\frac{s}{t}$ with $s,t \in S$ since $\frac{t}{s} \cdot \frac{s}{t} = 1$. Thus $\iota_S^{-1}(\mathfrak{q})$ intersects S trivially. This shows that Ψ is well-defined.

We continue with the proof that $\Psi \circ \Phi$ is the identity, i.e. $\iota_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A that does not intersect S. The inclusion $\mathfrak{p} \subset \iota_S^{-1}(S^{-1}\mathfrak{p})$ is trivial. The reverse inclusion can be shown as follows. The set $\iota_S^{-1}(S^{-1}\mathfrak{p})$ consists of all elements $a \in A$ such that $\frac{a}{1} = \frac{b}{s}$ for some $b \in \mathfrak{p}$ and $s \in S$. This equation says that there is a $t \in S$ such that $tsa = tb$. Since $b \in \mathfrak{p}$, we have $tsa = tb \in \mathfrak{p}$. Since $ts \notin \mathfrak{p}$, we have $a \in \mathfrak{p}$, as desired.

We continue with the proof that $\Phi \circ \Psi$ is the identity, i.e. $S^{-1}\iota_S^{-1}(\mathfrak{q}) = \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of $S^{-1}A$. The inclusion $S^{-1}\iota_S^{-1}(\mathfrak{q}) \subset \mathfrak{q}$ is trivial. The reverse inclusion can be shown as follows. Let $\frac{a}{s} \in \mathfrak{q}$. Then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in \mathfrak{q}$ and $a \in \iota_S^{-1}\mathfrak{q}$. Thus $\frac{a}{s} \in S^{-1}\iota_S^{-1}(\mathfrak{q})$, as desired. This concludes the proof of the proposition. □

Residue fields

Let A be a monoid with zero, $\mathfrak{p}$ a prime ideal of A and $S = A - \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the complement of the units of $S^{-1}A$ and therefore the unique maximal ideal of $S^{-1}A$.

Definition 3.6.5. Let A be a monoid with zero and $\mathfrak{p}$ a prime ideal of A. The *residue field at* $\mathfrak{p}$ is the monoid with zero $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{c}(S^{-1}\mathfrak{p})$ where S is the complement of $\mathfrak{p}$ in A and $\mathfrak{c}(S^{-1}\mathfrak{p})$ is the congruence on $A_\mathfrak{p}$ that is generated by $S^{-1}\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of A. Then the residue field at $\mathfrak{p}$ comes with a canonical morphism $A \to k(\mathfrak{p})$, which is the composition of the localization map $A \to A_\mathfrak{p}$ with the quotient map $A_\mathfrak{p} \to k(\mathfrak{p})$.

Corollary 3.6.6. *Let A be a monoid with zero, $\mathfrak{p}$ a prime ideal of A and $S = A - \mathfrak{p}$. Then $k(\mathfrak{p})$ is nontrivial and $k(\mathfrak{p})^\times = k(\mathfrak{p}) - \{0\}$.*

Proof. Note that $\mathfrak{p}$ is the maximal prime ideal that does not intersect S. By Proposition 3.6.4, $\mathfrak{m} = S^{-1}\mathfrak{p}$ is the unique maximal of $S^{-1}A$. Thus the kernel of $S^{-1}A \to k(\mathfrak{p})$ is $\mathfrak{m}$, which shows that $k(\mathfrak{p})$ is nontrivial. Since $(S^{-1}A)^\times = S^{-1}A - \mathfrak{m}$, we see that $(S^{-1}A)^\times \to k(\mathfrak{p}) - \{0\}$ is surjective, which shows that all nonzero elements of $k(\mathfrak{p})$ are invertible. □

Chapter 4

Blueprints

A blueprint can be described as a hybrid of a monoid with zero and a semiring. Blueprints continue sharing certain properties with rings in the same way as monoids and semirings do, but in other aspect the deviation from rings increases. In this section, we will discuss the aspects of blueprints that will be relevant for this text.

Blueprints were first introduced by the author in [Lor12]. Note that the definition of a blueprint in these notes is more restrictive than the original definition. Namely, the definition that we use in this text, as most other sources on blueprints do, correspond to proper blueprints with zero in [Lor12]. As a complementary reading to this chapter, the reader might want to consider the overview papers [Lor16] and [Lor18].

4.1 The category of blueprints

Definition 4.1.1. A *blueprint* is a pair $B = (B^\bullet, B^+)$ of a semiring B^+ and a multiplicative subset $B^\bullet$ of B^+ that contains 0 and spans B^+ as a semiring. A *morphism of blueprints* $f : B \to C$ is a semiring morphism $f^+ : B^+ \to C^+$ with $f(B^\bullet) \subset C^\bullet$. We denote the restriction of f^+ to the respective multiplicative subsets by $f^\bullet : B^\bullet \to C^\bullet$. We denote the category of blueprints by Blpr.

Let $B = (B^\bullet, B^+)$ be a blueprint. The *ambient semiring of* B is B^+ and the *underlying monoid of* B is $B^\bullet$. We write $a \in B$ for $a \in B^\bullet$ and $S \subset B$ for $S \subset B^\bullet$. The *unit group of* B is $B^\times = (B^\bullet)^\times$. A *blue field* is a blueprint B with $B^\times = B - \{0\}$.

Note that this definition yields tautologically a functor $(-)^+ : \mathrm{Blpr} \to \mathrm{SRings}$. Note further that the underlying monoid $B^\bullet$ of a blueprint B is a monoid with zero and given a morphism of blueprints $f : B \to C$, the map $f^\bullet : B^\bullet \to C^\bullet$ is a morphism of monoids with zero. Thus we obtain a functor $(-)^\bullet : \mathrm{Blpr} \to \mathrm{Mon}$.

Finally note that a morphism $f : B \to C$ of blueprints is already determined by $f^\bullet : B^\bullet \to C^\bullet$ since $B^\bullet$ spans B^+ as a semiring. This allows us to describe a morphism $f : B \to C$ of blueprints in terms of the monoid morphism $f^\bullet : B^\bullet \to C^\bullet$.

Example 4.1.2. Some first examples of blueprints are the following:

- $\{0,1\} \subset \mathbb{N}$, which is an initial object of Blpr;
- $\{0\} \subset \{0\}$, which is a terminal object of Blpr;
- $\{0,\pm 1\} \subset \mathbb{Z}$;
- $[0,1] \subset \mathbb{R}_{\geqslant 0}$;

- $\{aT_1^{e1} \cdots T_n^{e_n}\}_{a \in R, e_1, \ldots, e_n \in \mathbb{N}} \subset R[T_1, \ldots, T_n]$ where R is a semiring.

Note that we will denote $(\{0,1\}, \mathbb{N})$ by $\mathbb{F}_1$, cf. section 4.2, $(\{0, \pm 1\}, \mathbb{Z})$ by $\mathbb{F}_{1^2}$, cf. Example 4.4.4 and the last blueprint of this list by $R^{\text{blue}}[T_1, \ldots, T_n]$, cf. section 4.3.

Definition 4.1.3. Let B be a blueprint. A *B-algebra* is a blueprint C together with a blueprint morphism $\iota_C : B \to C$. Often we write only C for a B-algebra without mentioning ι_C explicitly. A *morphism between B-algebras C and D* or a *B-linear morphism* is a blueprint morphism $f : C \to D$ such that $\iota_D = f \circ \iota_C$. This defines the category Alg_B of B-algebras. We denote the sets of B-linear morphisms from C to D by $\text{Hom}_B(C, D)$.

Let C and D be B-algebras. Then there is a morphism $\alpha : C^\bullet \otimes_{B^\bullet} D^\bullet \to (C^+ \otimes_{B^+} D^+)^\bullet$ of monoids with zero that sends $c \otimes d$ to $c \otimes d$. We define the *tensor product of C and D over B* as the blueprint $C \otimes_B D = (\operatorname{im} \alpha^\bullet, C^+ \otimes_{B^+} D^+)$.

Exercise 4.1.4 (Tensor products)**.** Show that $C \otimes_B D$ is indeed a blueprint. Describe the canonical inclusions $C \to C \otimes_B D$ and $D \to C \otimes_B D$. Formulate and prove the usual properties of the tensor product: (1) The tensor product is the colimit (or pushout) of the diagram $C \xleftarrow{\iota_C} C \xrightarrow{\iota_D} D$; (2) every B-bilinear morphism from $C \times D$ defines a unique morphism from $C \otimes_B D$.

Let $f : B \to C$ be a blueprint morphism. Then the precomposition with f defines a functor $\text{Alg}_C \to \text{Alg}_B$, which is called the *restriction of scalars*. Show that $- \otimes_B C$ defines a left adjoint $\text{Alg}_B \to \text{Alg}_C$ to the restriction of scalars.

Exercise 4.1.5 (Limits and colimits)**.** Show that the category of blueprints is complete and cocomplete. More precisely, show the following assertions.

Let $\{B_i\}$ be a family of blueprints. Then there is a canonical morphism $\alpha^+ : (\prod B_i^\bullet)^+ \to \prod B_i^+$ of semirings. Define $\prod B_i = (\prod B_i^\bullet, \operatorname{im} \alpha^+)$ and describe the canonical projections $\pi_j : \prod B_i \to B_j$. Show that $\prod B_i$ is a product of the B_i.

Similarly, there is a canonical morphism $\alpha^\bullet : \bigotimes B_i^\bullet \to (\bigotimes B_i^+)^\bullet$ of monoids with zero. Define $\bigotimes B_i = (\operatorname{im} \alpha^\bullet, \bigotimes B_i^+)$ and describe the canonical inclusions $\iota_j : B_j \to \bigotimes B_i$. Show that $\bigotimes B_i$ is a coproduct of the B_i.

Let $f, g : B \to C$ be two blueprint morphisms. Then there is a canonical morphism $\alpha^+ : \operatorname{eq}(f^\bullet, g^\bullet)^+ \to \operatorname{eq}(f^+, g^+)$ of semirings. Define $\operatorname{eq}(f, g) = (\operatorname{eq}(f^\bullet, g^\bullet), \operatorname{im} \alpha^+)$, which comes with a canonical inclusion $\operatorname{eq}(f, g) \to B$. Show that $\operatorname{eq}(f, g)$ is an equalizer of f and g.

Similarly there is a canonical morphism $\alpha^\bullet : \operatorname{coeq}(f^\bullet, g^\bullet) \to \operatorname{coeq}(f^+, g^+)^\bullet$ of monoids with zero. Define $\operatorname{coeq}(f, g) = (\operatorname{im} \alpha^\bullet, \operatorname{coeq}(f^+, g^+))$, which comes with a canonical projection $C \to \operatorname{coeq}(f, g)$. Show that $\operatorname{coeq}(f, g)$ is a coequalizer of f and g.

Exercise 4.1.6 (Monomorphisms, isomorphisms and epimorphisms)**.** Let $f : B \to C$ be a blueprint morphism. Show that f is a monomorphism if and only if $f^\bullet$ is injective; f is an isomorphism if and only if both $f^\bullet$ and f^+ are bijective; f is an epimorphisms if f^+ is surjective. Give an example of an epimorphism f for which f^+ is not surjective.

Exercise 4.1.7 (Axiomatic blueprints)**.** There is a different but equivalent approach to blueprints. This alternative viewpoint has been used in previous texts about blueprints, as in [Lor12] and [Lor16]. In this exercise, we explain the connection to this alternative definition.

We define an *axiomatic blueprint* as a pair $B = (A, \mathcal{R})$ of a monoid with zero A together with a *preaddition* $\mathcal{R}$, which is an equivalence relation on $\mathbb{N}[A]$ that satisfies for all $x, y, z, t \in \mathbb{N}[A]$ and $a, b \in A$ that

(1) $x \equiv y$ and $z \equiv t$ implies $x + z \equiv y + t$ and $xz \equiv yt$,

(2) $a \equiv b$ implies $a = b$ as elements of A, and

(3) $0_A \equiv 0_{\mathbb{N}[A]}$, i.e. the zero of A is equivalent to zero of $\mathbb{N}[A]$,

where we write $x \equiv y$ for $(x,y) \in \mathcal{R}$. We also write $B^\bullet$ for A and say that $x \equiv y$ holds in B if $(x,y) \in \mathcal{R}$. A *morphism between axiomatic blueprints* B_1 *and* B_2 is a morphism $f : B_1^\bullet \to B_2^\bullet$ of monoids with zero such that for all $a_i, b_j \in B_1^\bullet$ with $\sum a_i \equiv \sum b_j$ in B_1, we have $\sum f(a_i) \equiv \sum f(b_j)$ in B_2.

Let $B = (A, \mathcal{R})$ be an axiomatic blueprint. Show that $\mathcal{R}$ is a congruence on $\mathbb{N}[A]$ and denote the semiring $\mathbb{N}[A]/\mathcal{R}$ by B^+. Show that the natural map $A \to \mathbb{N}[A] \to B^+$ is injective and defines a blueprint $(B^\bullet, B^+)$. Conversely, we can associate with a blueprint $(B^\bullet, B^+)$ the axiomatic blueprint $(B^\bullet, \mathcal{R})$ where $\mathcal{R}$ is the congruence kernel of the quotient map $\mathbb{N}[B^\bullet] \to B^+$.

Show that every morphism $f : B_1 \to B_2$ of axiomatic blueprints induces a semiring morphism $f^+ : B_1^+ \to B_2^+$, which satisfies $f^+(B_1^\bullet) \subset B_2^\bullet$. Show that this defines an equivalence between the category of axiomatic blueprints with Blpr.

Basic facts about reflective subcategories

In the following sections, we will encounter several reflective and coreflective subcategories of Blpr. The following exercises contain the definition of a (co)reflective category and discuss its main properties. Though reflective subcategories is a standard topic in category theory, most expositions are either incomplete or use more advanced results from category theory than is necessary for our purposes. Accessible references are sections 3.4 and 3.5 in Borceux's book [Bor94] and section IV.3 in MacLane's book [Mac71].

Exercise 4.1.8. Let $\mathcal{C}$ be a category. A *reflective subcategory of* $\mathcal{C}$ is a full subcategory $\mathcal{D}$ such that the inclusion functor $\iota : \mathcal{D} \to \mathcal{C}$ has a left adjoint $\rho : \mathcal{C} \to \mathcal{D}$, i.e. there are bijections $\Phi : \mathrm{Hom}_{\mathcal{C}}(C, \iota(D)) \to \mathrm{Hom}_{\mathcal{D}}(\rho(C), D)$ for every C in $\mathcal{C}$ and D in $\mathcal{D}$ that are functorial in C and D. The functor ρ is called a *reflection of* $\mathcal{C}$ *in* $\mathcal{D}$.

Show that $\rho \circ \iota$ is isomorphic to the identity functor on $\mathcal{D}$. More precisely, show that the counit of the adjunction $\epsilon_D = \Phi(\mathrm{id}_{\iota(D)}) : \rho \circ \iota(D) \to D$ is an isomorphism for every D in $\mathcal{D}$. In other words, this shows that if ρ is a reflection of a full embedding $\iota : \mathcal{D} \to \mathcal{C}$ of categories, then ρ is a left inverse of ι.

Conversely, assume that $\iota : \mathcal{D} \to \mathcal{C}$ is an arbitrary functor of categories that has a left adjoint and left inverse $\rho : \mathcal{C} \to \mathcal{D}$. Show that ι is fully faithful and that the image of ι is a reflective subcategory of $\mathcal{C}$.

Exercise 4.1.9. Let $\mathcal{C}$ be a complete and cocomplete category and $\iota : \mathcal{D} \to \mathcal{C}$ a reflective subcategory with reflection ρ. Let Δ be a *diagram in* $\mathcal{D}$, i.e. a family of objects and morphisms in $\mathcal{D}$. Denote by $\iota(\Delta)$ the diagram in $\mathcal{C}$ that results from Δ by applying ι to each object and morphism in Δ.

Show that $\rho\big(\lim \iota(\Delta)\big)$ is a limit $\lim \Delta$ of Δ in $\mathcal{D}$ and that $\iota(\lim \Delta)$ is naturally isomorphic to $\lim \iota(\Delta)$. Show that $\rho\big(\mathrm{colim}\, \iota(\Delta)\big)$ is a colimit $\mathrm{colim}\, \Delta$ of Δ in $\mathcal{D}$. Find an example where the natural morphism $\mathrm{colim}\, \iota(\Delta) \to \iota(\mathrm{colim}\, \Delta)$ is not an isomorphism.

Exercise 4.1.10. Let $\mathcal{C}$ be a category. A *coreflective subcategory of* $\mathcal{C}$ is a full subcategory $\mathcal{D}$ such that the inclusion functor $\iota : \mathcal{D} \to \mathcal{C}$ has a right adjoint $\rho : \mathcal{C} \to \mathcal{D}$. Formulate and prove the analogous properties from Exercises 4.1.8 and 4.1.9.

4.2 Semirings and monoids as blueprints

Let R be a semiring. Then we define the associated blueprint as $R^{\text{blue}} = (R,R)$, thus $(R^{\text{blue}})^\bullet = (R^{\text{blue}})^+ = R$. Every morphism $f : R \to S$ of semirings is tautologically a morphism of blueprints, which we denote by $f^{\text{blue}} : R^{\text{blue}} \to S^{\text{blue}}$. This yields a functor

$$(-)^{\text{blue}} : \text{SRings} \longrightarrow \text{Blpr}.$$

Lemma 4.2.1. *The functor* $(-)^+ : \text{Blpr} \to \text{SRings}$ *is a left adjoint and left inverse to* $(-)^{\text{blue}} : \text{SRings} \to \text{Blpr}$. *Thus we can identify* SRings *with a reflective subcategory of* Blpr.

Proof. By its very definition, it is clear that $(-)^+$ is a left inverse to $(-)^{\text{blue}}$. Let B be a blueprint and R a semiring. A blueprint morphism $f : B \to R^{\text{blue}}$ is a semiring morphism $f^+ : B^+ \to (R^{\text{blue}})^+$ such that $f^+(B^\bullet) \subset (R^{\text{blue}})^\bullet = R$. Since the latter condition is vacuous, we obtain a natural bijection $\text{Hom}(B, R^{\text{blue}}) \to \text{Hom}(B^+, R)$, which shows that $(-)^+$ is a left adjoint to $(-)^{\text{blue}}$. □

This allows us to consider any semiring as a blueprint. In particular, we consider the natural numbers $\mathbb{N}$, the Boolean numbers $\mathbb{B}$, the tropical numbers $\mathbb{T}$ and their integers $\mathcal{O}_{\mathbb{T}}$, as well as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ as blueprints in the following.

Remark 4.2.2. We warn the reader at this point that coproducts and free algebras are not preserved by the inclusion $(-)^{\text{blue}} : \text{SRings} \to \text{Blpr}$. For instance if R is a semiring and S and T are two R-algebras, then $(S \otimes_R T)^{\text{blue}}$ is in general not isomorphic to $S^{\text{blue}} \otimes_{R^{\text{blue}}} T^{\text{blue}}$. But in accordance with Exercise 4.1.9, there is a canonical isomorphism $\big(S^{\text{blue}} \otimes_{R^{\text{blue}}} T^{\text{blue}}\big)^+ \to S \otimes_R T$ where the former tensor product is a tensor product of blueprints and the latter tensor product is a tensor product of semirings.

There is a similar discrepancy between the construction of free algebras; cf. section 4.3 for more details. To avoid confusion, we shall often add a symbol "$+$" to make clear that we refer to the corresponding construction in SRings; for instance, we write $S \otimes_R^+ T$ and $R[T_1, \dots, T_n]^+$.

Exercise 4.2.3. Show that Rings is a reflective subcategory of SRings. More precisely, show that $- \otimes_{\mathbb{N}} \mathbb{Z}$ is a left adjoint of the inclusion functor $\iota : \text{Rings} \to \text{SRings}$.

Conclude that the composition $(-)^{\text{blue}} \circ \iota : \text{Rings} \to \text{SRings} \to \text{Blpr}$ has a left adjoint and left adjoint ρ. Give an explicit description of ρ.

Let A be a monoid with zero 0_A and $A^+ = \mathbb{N}[A]/\mathfrak{c}(0_A)$ the associated semiring, cf. Exercise 3.2.6. Then we define the associated blueprint as $A^{\text{blue}} = (A, A^+)$. Given a morphism $f : A \to B$ of monoids with zeros, we obtain a morphism of semirings $f^+ : A^+ \to B^+$ by linear extension, cf. Exercise 3.2.6. We define $f^{\text{blue}} : A^{\text{blue}} \to B^{\text{blue}}$ as $f^+ : A^+ \to B^+$. This yields a functor

$$(-)^{\text{blue}} : \text{Mon} \longrightarrow \text{Blpr}.$$

Lemma 4.2.4. *The functor* $(-)^\bullet : \text{Blpr} \to \text{Mon}$ *is a right adjoint and left inverse of* $(-)^{\text{blue}} : \text{Mon} \to \text{Blpr}$. *Thus we can identify* Mon *with a coreflective subcategory of* Blpr.

Proof. Since we can recover a monoid with zero A from A^{blue} as $(A^{\text{blue}})^\bullet$ and a morphism $f : A \to B$ from f^{blue} as $f = (f^{\text{blue}})^\bullet$, we see that $(-)^\bullet$ is a left inverse of $(-)^{\text{blue}}$.

Let A be a monoid with zero and B a blueprint. A blueprint morphism $f : A^{\text{blue}} \to B$ determines a morphism $f^\bullet : A = (A^{\text{blue}})^\bullet \to B^\bullet$ of monoids with zero, and f is uniquely determined by $f^\bullet$. This defines an injection $\text{Hom}(A^{\text{blue}}, B) \to \text{Hom}(A, B^\bullet)$, which is a surjective since every morphism $g : A \to B^\bullet$ of monoids with zero extends to a semiring morphism $g^+ : A^+ \to B^+$. We conclude that $(-)^\bullet$ is a right adjoint of $(-)^{\text{blue}}$. $\square$

This allows us to consider every monoid as a blueprint, and we carry over the notation that we have used for monoids. In particular, we have $\mathbb{F}_1 = (\{0,1\},\mathbb{N})$.

In contrast to the situation of the inclusion SRings $\to$ Blpr, the inclusion Mon $\to$ Blpr preserves colimits and free algebras, but not limits. For example, the product $A \times B$ of two monoids with zeros A and B in Mon is evidently a monoid with zero. However, the product $A^{\text{blue}} \times B^{\text{blue}}$ in Blpr is the blueprint $(A \times B, A^+ \times B^+)$, and the semiring morphism $(A \times B)^+ \to A^+ \times B^+$ induced by the identity on $A \times B$ is not an isomorphism if A and B are nontrivial. For instance, the elements $(0_A, 1_B) + (1_A, 0_B)$ and $(1_A, 1_B)$ have the same image.

4.3 Free algebras

Definition 4.3.1. Let B be a blueprint and A a monoid with zero. The *monoid algebra of A over B* is the blueprint

$$B[A] \; = \; B \otimes_{\mathbb{F}_1} A^{\text{blue}} \; = \; \left(B^\bullet \otimes_{\mathbb{F}_1} A, B^+ \otimes^+_{\mathbb{N}} A^+\right).$$

Let $\{T_i\}_{i\in I}$ be a set. The *free algebra in $\{T_i\}_{i\in I}$ over B* is the blueprint $B[T_i]_{i\in I} = B[A]$ for $A = \mathbb{F}_1[T_i]_{i\in I}$. We write $B[T_1,\dots,T_n]$ if $I = \{1,\dots,n\}$.

Note that the monoid algebra $B[A]$ is a B-algebra with respect to the inclusion $B \to B[A]$ sending a to $a \otimes 1_A$. Note that if R is a semiring and $n \geqslant 1$, then the monoid algebra $R^{\text{blue}}[T_1,\dots,T_n]$ is not equal to the blueprint associated with the polynomial semiring $R[T_1,\dots,T_n]$. But there is a natural isomorphism $\left(R^{\text{blue}}[T_1,\dots,T_n]\right)^+ \to R[T_1,\dots,T_n]$, in accordance with Lemma 4.2.1 and Exercise 4.1.9.

Exercise 4.3.2. Formulate and prove the universal properties for $B[A]$ and $B[T_i]$.

Example 4.3.3. Let R be a semiring. As a blueprint, the free R-algebra in $T_1,\dots,T_n$ is

$$R[T_1,\dots,T_n] \; = \; \left(\{aT_1^{e_1}\cdots T_n^{e_n}\}_{a\in R, e_1,\dots,e_n\in\mathbb{N}}, R[T_1,\dots,T_n]^+\right).$$

Exercise 4.3.4. Show that $\mathbb{F}_1[T_1,\dots,T_n]^+ = \mathbb{N}[T_1,\dots,T_n]^+$.

4.4 Quotients and congruences

Definition 4.4.1. Let B be a blueprint. A *congruence on B* is a congruence on the ambient semiring B^+. Let $\mathfrak{c}$ be a congruence on B and $\pi : B^+ \to B^+/\mathfrak{c}$ the quotient map. The *quotient of B by $\mathfrak{c}$* is the blueprint $B/\!\!/\mathfrak{c} = (\pi(B^\bullet), B^+/\mathfrak{c})$.

The congruence kernel of a blueprint morphism $f : B \to C$ is the congruence kernel $\mathfrak{c}(f^+)$ of the semiring morphism $f^+ : B^+ \to C^+$. A *quotient of a blueprint B* is a class of surjective blueprint morphisms $f : B \to C$, i.e. $f(B^\bullet) = C^\bullet$, where two surjection $f : B \to C$ and $f' : B \to C'$ are equivalent if there is an isomorphism $g : C \to C'$ such that $f' = g \circ f$.

Given a blueprint B and a subset $S = \{(x_i, y_i)\}$ of $B^+ \times B^+$, we denote by $\langle S \rangle = \langle x_i \equiv y_i \rangle$ the congruence on B^+ generated by S.

Note that $B/\!\!/\mathfrak{c}$ is indeed a blueprint: by Proposition 2.4.4, $B^+/\mathfrak{c}$ is a semiring; it is obvious that $\pi(B^\bullet)$ is a multiplicative subset of $B^+/\mathfrak{c}$ that contains 0 and 1 and spans $B^+/\mathfrak{c}$. Note further that the quotient map $\pi : B \to B/\!\!/\mathfrak{c}$ is a morphism of blueprints, which satisfies $\pi^+(x) = \pi^+(y)$ whenever $x \sim_\mathfrak{c} y$. By Lemma 2.4.7, the congruence kernel of a blueprint morphism $f : B \to C$ is a congruence on B.

Proposition 4.4.2. *Let $S = \{(x_i, y_i)\}$ be a subset of $B^+ \times B^+$ and $\mathfrak{c} = \langle S \rangle$ the congruence generated by S. Let $\pi : B \to B/\!\!/\mathfrak{c}$ be the quotient map. Given a morphism $f : B \to C$ such that $f(x_i) = f(y_i)$ for all i, there is a unique morphism $\bar{f} : B/\!\!/\mathfrak{c} \to C$ such that $f = \bar{f} \circ \pi$.*

Proof. By Lemma 2.4.8, there is a unique semiring morphism $g : (B/\!\!/\mathfrak{c})^+ = B^+/\mathfrak{c} \to C$ such that $f^+ = g \circ \pi^+$. Since $g\big((B/\!\!/\mathfrak{c})^\bullet\big) = f(B^\bullet)$ and $f(B^\bullet) \subset C^\bullet$, the semiring morphism g defines a blueprint morphism $\bar{f} : B/\!\!/\mathfrak{c} \to C$ with $\bar{f}^+ = g$ that satisfies $f = \bar{f} \circ \pi$. □

Proposition 4.4.3. *The associations*

$$\begin{array}{ccc}
\{\textit{congruences on } B\} & \stackrel{1:1}{\longleftrightarrow} & \{\textit{quotients of } B\} \\
\mathfrak{c} & \stackrel{\Phi}{\longmapsto} & B \to B/\!\!/\mathfrak{c} \\
\mathfrak{c}(\pi) & \stackrel{\Psi}{\longleftarrow\!\shortmid} & \pi : B \twoheadrightarrow C
\end{array}$$

are mutually inverse bijections.

Proof. It is clear that $\mathfrak{c}$ is the congruence kernel of $B \to B/\!\!/\mathfrak{c}$. If $\pi : B \to C$ is a surjective morphism of blueprints, then $\pi^+ : B^+ \to C^+$ is surjective and $C^\bullet = \pi^+(B^\bullet)$. Thus $C = B/\!\!/\mathfrak{c}$ where $\mathfrak{c}$ is the congruence kernel of π. □

The free algebra construction and the characterization of quotients of blueprints by congruences allows for a convenient notation for blueprints: given any blueprint B, e.g. a monoid or a semiring, and any subset $\{(f_i, g_i)\}$ of $B[T_1, \ldots, T_n]^+ \times B[T_1, \ldots, T_n]^+$, we can define the blueprint $B[T_1, \ldots, T_n]/\!\!/\langle f_i \equiv g_j \rangle$.

Example 4.4.4 (Cyclotomic extensions of $\mathbb{F}_1$)**.** Let μ_n be a cyclic group of order n with generator ζ_n. The *n-th cyclotomic extension of* $\mathbb{F}_1$ is the blueprint

$$\mathbb{F}_{1^n} \;=\; \mathbb{F}_1[\mu_n]/\!\!/\langle \textstyle\sum_{i=1}^{n/d} \zeta_n^{di} \,|\, d < n \text{ is a divisor of } n \rangle.$$

For $n \geqslant 2$, we can identify ζ_n with a primitive n-th root of unity in the cyclotomic number field $\mathbb{Q}[\zeta_n]$, which yields an isomorphism of the ambient semiring $\mathbb{F}_{1^n}^+$ with the ring of integers of $\mathbb{Q}[\zeta_n]$. For $n = 1$, we have $\mathbb{F}_{1^1} = \mathbb{F}_1$ and for $n = 2$, we have that $-1 = \zeta_2$ is an additive inverse of 1 and $\mathbb{F}_{1^2} = \{0, \pm 1\}/\!\!/\langle 1 + (-1) \equiv 0 \rangle$.

Exercise 4.4.5. Let $n \geqslant 1$ and k a field. If the characteristic of k does not divide n, then assume that k contains all n-th roots of unity. If k is of finite characteristic p and p^k is the highest power of p dividing n, i.e. $m = n/p^k$ is not divisible by p, then assume that k contains all m-th roots of unity. Show that there exists a morphism $\mathbb{F}_{1^n} \to k$ of blueprints and that for any two morphisms $f, g : \mathbb{F}_{1^n} \to k$, there is an automorphism $\sigma : k \to k$ and a unique automorphism $\tau : \mathbb{F}_{1^n} \to \mathbb{F}_{1^n}$ such that $g = \sigma \circ f = f \circ \tau$.

Show that this defines a group morphism $\mathrm{Aut}(k) \to \mathrm{Aut}(\mathbb{F}_{1^n})$ from the group of field automorphism of k to the group of blueprint automorphisms of $\mathbb{F}_{1^n}$, which sends σ to τ. Determine the image G of $\mathrm{Aut}(k) \to \mathrm{Aut}(\mathbb{F}_{1^n})$ and the fixed blue subfield of $\mathbb{F}_{1^n}$ by G.

Let n be even. Show that $\mathbb{F}_{1^2}$ is fixed by every automorphism of $\mathbb{F}_{1^n}$. Define $\mathrm{Gal}(\mathbb{F}_{1^n}/\mathbb{F}_{1^2}) = \mathrm{Aut}(\mathbb{F}_{1^n})$. Show that there is a Galois correspondence between the subgroups of $\mathrm{Gal}(\mathbb{F}_{1^n}/\mathbb{F}_{1^2})$ and the blue subfields of $\mathbb{F}_{1^n}$ containing $\mathbb{F}_{1^2}$.

Example 4.4.6. Let k be a ring and R be a k-algebra, i.e. a ring homomorphism $k \to R$. A representation $R \simeq k[T_1, \dots, T_n]^+/I$ defines the associated blueprint

$$k[T_1,\dots,T_n]/\!\!/\langle x \equiv y | x-y \in I\rangle \;=\; \big(\big\{[aT_1^{e_1}\cdots T_n^{e_n}]\big\}, R\big)$$

where $[aT_1^{e_1}\cdots T_n^{e_n}]$ is the class of $aT_1^{e_1}\cdots T_n^{e_n} \in k[T_1,\dots,T_n]$ in $R = k[T_1,\dots,T_n]/I$.

Exercise 4.4.7. Let $B = \mathbb{F}_1[T_1,\dots,T_4]/\!\!/\langle T_1T_4 \equiv T_2T_3 + 1\rangle$. Describe a bijection of the morphism set $\mathrm{Hom}(B,\mathbb{N})$ with the set of 2×2-matrices with coefficients in $\mathbb{N}$ and determinant 1.

Exercise 4.4.8. Show that $\mathbb{F}_1[T_2,T_{-2}]/\!\!/\langle T_2 \equiv 1+1, T_2+T_{-2} \equiv 0\rangle$ is isomorphic to $(2\mathbb{Z}\cup\{1\},\mathbb{Z})$. Find a representation of $(\mathbb{Z},\mathbb{Z})$ as $A/\!\!/\mathfrak{c}$ where A is a monoid with zero and $\mathfrak{c}$ a congruence on A^+.

Exercise 4.4.9. Let B be a blueprint, $\mathfrak{c}$ a congruence on B and $\pi : B \to B/\!\!/\mathfrak{c}$ the quotient map. Show that the restriction $\mathfrak{c}^\bullet$ of $\mathfrak{c}$ to $B^\bullet$ is a congruence on the monoid with zero $B^\bullet$ and that $(B/\!\!/\mathfrak{c})^\bullet \simeq B^\bullet/\mathfrak{c}^\bullet$. Conversely, show that every congruence $\mathfrak{c}^\bullet$ on the underlying monoid $B^\bullet$ determines a congruence $\mathfrak{c}$ on B that is minimal among all congruences on B whose restriction to $B^\bullet$ is $\mathfrak{c}^\bullet$.

Conclude that a congruence $\mathfrak{c}$ on a blueprint B is the same as a pair $(\mathfrak{c}^\bullet,\mathfrak{c}^+)$ of a congruence $\mathfrak{c}^\bullet$ on the underlying monoid $B^\bullet$ and a congruence $\mathfrak{c}^+$ on the ambient semiring B^+ such that the inclusion $B^\bullet \to B^+$ induces an injection $B^\bullet/\mathfrak{c}^\bullet \hookrightarrow B^+/\mathfrak{c}^+$. In so far, we obtain the following picture:

$$\big\{\text{congruences on } B^\bullet\big\} \underset{\longrightarrow}{\twoheadleftarrow} \big\{\text{congruences on } B\big\} \xleftrightarrow{1:1} \big\{\text{congruences on } B^+\big\}$$

4.5 Reflective subcategories

The following properties characterize important subclasses of blueprints.

Definition 4.5.1. A blueprint B is

- *without zero divisors* if $B^\bullet$ is without zero divisors;
- *integral* (or *multiplicatively cancellative*) if $B^\bullet$ is integral;
- *with (additive) inverses* or *with* -1 if $B^\bullet$ contains an element -1 that is an additive inverse of 1 in B^+;
- *(additively) cancellative* if B^+ is cancellative;
- *(additively) idempotent* if B^+ is idempotent;

Lemma 4.5.2. *Let B be a blueprint.*

(1) If $0 = 1$ in B, then B is trivial, i.e. $B^\bullet = B^+ = \{0\}$.

(2) If B is integral, then B is without zero divisors.

(3) B is cancellative if and only if B^+ embeds into a ring.

(4) *If B is with -1, then B^+ is a ring. In particular, B is cancellative. Moreover, $(-1)^2 = 1$ and $-a = (-1) \cdot a$ is an additive inverse of a for every $a \in B$.*

(5) *B is with -1 if and only if there is a morphism $\mathbb{F}_{1^2} \to B$. The morphism $\mathbb{F}_{1^2} \to B$ is unique.*

(6) *B is idempotent if and only if there is a morphism $\mathbb{B} \to B$. The morphism $\mathbb{B} \to B$ is unique.*

(7) *If B is idempotent and cancellative, then it is trivial.*

Proof. Parts (1), (3) and (7) follow from the corresponding statements for rings, cf. Lemma 2.2.3 and Exercise 2.2.5. Part (2) follows from the corresponding fact for monoids with zeros, cf. Lemma 3.1.5.

We continue with (4). The semiring B^+ is a ring since every element $a \in B$ has an additive inverse, namely $(-1) \cdot a$. Clearly, a ring a cancellative. Multiplication of $1 + (-1) = 0$ by any element a of B yields $a + (-a) = 0$, which shows that $-a$ is an additive inverse of a. In particular, we get $(-1) + (-1)^2 = 0$ for $a = -1$. Thus $(-1)^2 = (-1)^2 + (-1) + 1 = 1$.

We continue with (5). If B is with -1, then B^+ is a ring by (4). Thus there exists a unique morphism $f : \mathbb{Z} \to B^+$. Since $-1 \in B^\bullet$, we have $f(\{0, \pm 1\} \subset B^\bullet$, which shows that f is a blueprint morphism $\mathbb{F}_{1^2} \to B$. Conversely, assume that there exists a morphism $f : \mathbb{F}_{1^2} \to B$. Then the semiring morphism $f^+ : \mathbb{Z} \to B^+$ maps -1 to the additive inverse -1 of 1 in B^+, and $-1 = f^\bullet(-1) \in B^\bullet$. This shows the first statement of (5). The second claim follows since the image of 1 determines the semiring morphism $\mathbb{F}_{1^2}^+ = \mathbb{Z} \to B^+$ uniquely.

We continue with (6). Assume that B is idempotent, i.e. $1 + 1 = 1$ in B^+. Then the unique morphism $\mathbb{F}_1 \to B$ factors through $\mathbb{B} = \mathbb{F}_1 /\!\!/ \langle 1 + 1 \equiv 1 \rangle$ by the universal property of the quotient, cf. Proposition 4.4.2. Conversely, assume that there is a morphism $f : \mathbb{B} \to B$. Then $1 + 1 = f^+(1) + f^+(1) = f^+(1+1) = f^+(1) = 1$, which shows that B is idempotent. Since the images of 0 and 1 are fixed, it is clear that $\mathbb{B} \to B$ is unique. This completes the proof of the lemma. □

Example 4.5.3. The cyclotomic extension $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$ is with -1 if and only if n is even; cf. Example 4.4.4 for the definition of $\mathbb{F}_{1^n}$. Its ambient semiring $\mathbb{F}_{1^n}^+$ is a ring for all $n \geqslant 2$. This shows that it is not true in general that a blueprint B is with -1 if B^+ is a ring. Another counterexample is the blueprint $B = (2\mathbb{Z} \cup \{1\}, \mathbb{Z})$ from Exercise 4.4.8.

Blueprints with inverses as a reflective subcategory

Let $\mathrm{Blpr}^{\mathrm{inv}} \subset \mathrm{Blpr}$ be the full subcategory of blueprints with inverses.

Lemma 4.5.4. *The category* $\mathrm{Blpr}^{\mathrm{inv}}$ *of blueprints with inverses is a reflective subcategory of* Blpr *with reflection* $(-)^{\mathrm{inv}} = - \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$.

Proof. Let B be a blueprint and C a blueprint with inverses. Note that $B^{\mathrm{inv}} = B \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$ is indeed a blueprint with inverses since there exists a morphism $\mathbb{F}_{1^2} \to B \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$, which sends a to $1 \otimes a$, cf. Lemma 4.5.2, part (5). Thus $(-)^{\mathrm{inv}}$ is well-defined.

The morphism $\iota : B \to B \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$ that sends a to $a \otimes 1$ induces a map $\Phi : \mathrm{Hom}(B^{\mathrm{inv}}, C) \to \mathrm{Hom}(B, C)$, which is functorial in B and C. By Lemma 4.5.2, part (5), there is a unique morphism $\mathbb{F}_{1^2} \to C$. Since $\mathbb{F}_1$ is initial, there are unique morphisms $\mathbb{F}_1 \to \mathbb{F}_{1^2}$ and $\mathbb{F}_1 \to B$, and the compositions $\mathbb{F}_1 \to B \to C$ and $\mathbb{F}_1 \to \mathbb{F}_{1^2} \to C$ are equal. By the universal property of the tensor product, g factors uniquely into $\iota \circ f$ for some morphism $f : B \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2} \to C$. This shows that Φ is a bijection and that $- \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$ is a left adjoint to the embedding $\mathrm{Blpr}^{\mathrm{inv}} \to \mathrm{Blpr}$. □

Cancellative blueprints as a reflective subcategory

Let $\mathrm{Blpr}^{\mathrm{canc}} \subset \mathrm{Blpr}$ be the full subcategory of cancellative blueprints.

Lemma 4.5.5. *The category* $\mathrm{Blpr}^{\mathrm{canc}}$ *of cancellative blueprints is a reflective subcategory of* Blpr *whose reflection* $(-)^{\mathrm{canc}} : \mathrm{Blpr} \to \mathrm{Blpr}^{\mathrm{canc}}$ *sends a blueprint B to* $B^{\mathrm{canc}} = B /\!\!/ \mathfrak{c}^{\mathrm{canc}}$ *where*

$$\mathfrak{c}^{\mathrm{canc}} = \big\langle x \equiv y \,\big|\, x+z = y+z \text{ for some } z \in B^+ \big\rangle.$$

Proof. To begin with, we show that $\mathfrak{c}^{\mathrm{canc}}$ is equal to $\mathfrak{c} = \{x \equiv y | x+z = y+z \text{ for some } z \in B^+\}$, i.e. that $\mathfrak{c}$ is a congruence. It is clear that $\mathfrak{c}$ is reflective and symmetric. For transitivity, consider $x \sim_{\mathfrak{c}} y$ and $y \sim_{\mathfrak{c}} z$, i.e. $x+r = y+r$ and $y+s = z+s$ for some $r,s \in B^+$. Then $x+r+s = y+r+s = z+r+s$ and $x \sim_{\mathfrak{c}} z$. For additivity and multiplicativity, consider $x \sim_{\mathfrak{c}} y$ and $z \sim_{\mathfrak{c}} t$, i.e. $x+r = y+r$ and $z+s = t+s$ for some $r,s \in B^+$. Then $x+z+r+s = y+t+r+s$ implies $x+z \sim_{\mathfrak{c}} y+t$. Using

$$ys+rs = (y+r)s = (x+r)s = xs+rs \quad \text{and} \quad rt+rs = r(t+s) = r(z+s) = rz+rs,$$

we get

$$xz+xs+rz+rs = (x+r)(z+s) = (y+r)(t+s) = yt+ys+rt+rs = xz+xs+rz+rs,$$

which implies $xz \sim_{\mathfrak{c}} yt$. Thus $\mathfrak{c}$ is a congruence on B and $\mathfrak{c}^{\mathrm{canc}} = \mathfrak{c}$. Moreover, we conclude that $\pi : B \to B^{\mathrm{canc}}$ is an isomorphism if B is cancellative.

We continue with showing that B^{canc} is a cancellative blueprint. Let $\pi : B \to B^{\mathrm{canc}}$ be the quotient map. Consider an equality $\pi(x)+\pi(r) = \pi(y)+\pi(r)$ in $(B^{\mathrm{canc}})^+$. Since $\mathfrak{c}^{\mathrm{canc}} = \mathfrak{c}$, we have $x+r+s = y+r+s$ for some $s \in B^+$. Thus $x \sim_{\mathfrak{c}} y$ and $\pi(x) = \pi(y)$. This shows that B^{canc} is cancellative and thus an object of $\mathrm{Blpr}^{\mathrm{canc}}$.

Let $f : B \to C$ be a morphism into a cancellative blueprint C and consider $x \sim_{\mathfrak{c}} y$, i.e. $x+z = y+z$ for some $z \in B^+$. Then $f^+(x)+f^+(z) = f^+(y)+f^+(z)$ and $f^+(x) = f^+(y)$ since C is cancellative. This shows that $\mathfrak{c}$ is contained in the congruence kernel of f. By the universal property of the quotient map $\pi : B \to B^{\mathrm{canc}}$, there is a unique morphism $f^{\mathrm{canc}} : B^{\mathrm{canc}} \to C$ such that $f = f^{\mathrm{canc}} \circ \pi$, cf. Proposition 4.4.2. Given an arbitrary morphism $f : B \to C$ of blueprints, we define $f^{\mathrm{canc}} = g^{\mathrm{canc}}$ where g is the composition $B \to C \to C^{\mathrm{canc}}$. This defines the functor $(-)^{\mathrm{canc}} : \mathrm{Blpr} \to \mathrm{Blpr}^{\mathrm{canc}}$.

Let C be a cancellative blueprint. Then the map $\mathrm{Hom}(B^{\mathrm{canc}}, C) \to \mathrm{Hom}(B, C)$ sending $f : B^{\mathrm{canc}} \to C$ to $f \circ \pi : B \to C$ is a bijection by what we have shown in the last paragraph. This shows that $(-)^{\mathrm{canc}}$ is a left adjoint to the embedding $\mathrm{Blpr}^{\mathrm{canc}} \to \mathrm{Blpr}$. $\square$

Idempotent blueprints as a reflective subcategory

Let $\mathrm{Blpr}^{\mathrm{idem}} \subset \mathrm{Blpr}$ be the full subcategory of idempotent blueprints.

Lemma 4.5.6. *The category* $\mathrm{Blpr}^{\mathrm{idem}}$ *of idempotent blueprints is a reflective subcategory of* Blpr *with reflection* $(-)^{\mathrm{idem}} = - \otimes_{\mathbb{F}_1} \mathbb{B}$.

Proof. Let B be a blueprint and C an idempotent blueprint. Note that $B^{\mathrm{idem}} = B \otimes_{\mathbb{F}_1} \mathbb{B}$ is indeed an idempotent blueprint since there exists a morphism $\mathbb{B} \to B \otimes_{\mathbb{F}_1} \mathbb{B}$, which sends a to $1 \otimes a$, cf. Lemma 4.5.2, part (6). Thus $(-)^{\mathrm{idem}}$ is well-defined.

The morphism $\iota : B \to B \otimes_{\mathbb{F}_1} \mathbb{B}$ that sends a to $a \otimes 1$ induces a map $\Phi : \mathrm{Hom}(B^{\mathrm{idem}}, C) \to \mathrm{Hom}(B, C)$, which is functorial in B and C. By Lemma 4.5.2, part (6), there is a unique morphism

$\mathbb{B} \to C$. Since $\mathbb{F}_1$ is initial, there are unique morphisms $\mathbb{F}_1 \to \mathbb{B}$ and $\mathbb{F}_1 \to B$, and the compositions $\mathbb{F}_1 \to B \to C$ and $\mathbb{F}_1 \to \mathbb{B} \to C$ are equal. By the universal property of the tensor product, g factors uniquely into $\iota \circ f$ for some morphism $f : B \otimes_{\mathbb{F}_1} \mathbb{B} \to C$. This shows that Φ is a bijection and that $- \otimes_{\mathbb{F}_1} \mathbb{B}$ is a left adjoint to the embedding $\mathrm{Blpr}^{\mathrm{idem}} \to \mathrm{Blpr}$. $\square$

We illustrate the subcategories considered so far in Figure 4.1 where a containment of areas in the illustration indicates a containment of the corresponding subcategories and an empty intersection of areas indicates that the trivial blueprint is the only object in common.

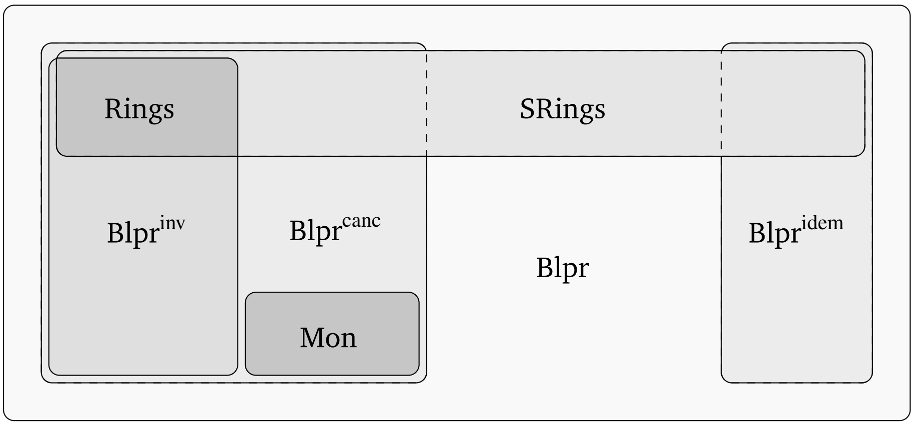

Figure 4.1: Some relevant subcategories of Blpr

Blue fields as a coreflective subcategory

Definition 4.5.7. Let B be a blueprint. The *unit field of B* is the subblueprint $B^\star$ of B whose underlying monoid is $(B^\star)^\bullet = \{0\} \cup B^\times$ and whose ambient semiring $(B^\star)^+$ is the subsemiring of B^+ generated by $(B^\star)^\bullet$.

Note that $B^\star$ is a blue field unless B is the trivial blueprint, which yields $B^\star = B = \{0\}$. Let $\mathrm{Blpr}^\star \subset \mathrm{Blpr}$ be the full subcategory whose objects are blue fields and the trivial blueprint.

Lemma 4.5.8. *The subcategory* $\mathrm{Blpr}^\star$ *is a coreflective subcategory of* Blpr *whose reflection sends a blueprint to its unit field.*

Proof. It is obvious that the unit field $B^\star$ of a blueprint is a blue field and thus in $\mathrm{Blpr}^\star$. Since every blueprint morphism $f : B \to C$ maps 0 to 0 and $B^\times$ to $C^\times$, we can restrict f to a morphism $f^\star : B^\star \to C^\star$ between the respective unit fields. This defines a functor $(-)^\star : \mathrm{Blpr} \to \mathrm{Blpr}^\star$.

For the same reason, the map $\mathrm{Hom}(B, C^\star) \to \mathrm{Hom}(B, C)$ that sends a morphism $f : B \to C$ from a blue field B to a blueprint C to its composition with the inclusion $C^\star \to C$ is a bijection. This shows that $(-)^\star$ is right adjoint to the inclusion of $\mathrm{Blpr}^\star$ into Blpr as a subcategory. $\square$

Remark 4.5.9. Lemma 4.5.8 stays in stark contrast to the analogous situation for (semi)fields and (semi)rings. The Lemma implies that $\mathrm{Blpr}^\star$ is complete and cocomplete and that the colimit of blue fields, calculated in Blpr, is again a blue field. A particular instance is that the tensor product of blue fields is again a blue field. Of course, this also follows directly from the construction of the tensor product as the set of pure tensors.

Partially additive blueprints as a coreflective subcategory

Exercise 4.5.10 (Partially additive blueprints)**.** A *partially additive blueprint* is a blueprint B such that the congruence kernel of the quotient map $(B^\bullet)^+ \to B^+$ is generated by relations of the form $a \sim \sum b_j$ with $a, b_j \in B^\bullet$. Let $\mathrm{Blpr}^{\mathrm{padd}}$ be the full subcategory of partially additive blueprints in Blpr.

Show that $\mathrm{Blpr}^{\mathrm{padd}}$ is a coreflective subcategory of Blpr whose reflection sends a blueprint B to $B^\bullet /\!\!/ \mathfrak{c}^{\mathrm{padd}}$ where

$$\mathfrak{c}^{\mathrm{padd}} \;=\; \big\langle\, a \equiv \textstyle\sum b_j \,|\, a, b_j \in A \text{ and } a = \sum b_j \text{ in } B^+ \,\big\rangle.$$

Show that $\mathrm{Blpr}^{\mathrm{padd}}$ contains Mon, SRings and $\mathrm{Blpr}^{\mathrm{inv}}$. Show that there exists nontrivial blueprints in the intersections of $\mathrm{Blpr}^{\mathrm{padd}}$ with $\mathrm{Blpr}^{\mathrm{canc}}$ and with $\mathrm{Blpr}^{\mathrm{idem}}$, but that $\mathrm{Blpr}^{\mathrm{padd}}$ neither contains nor is contained in either of $\mathrm{Blpr}^{\mathrm{canc}}$ and $\mathrm{Blpr}^{\mathrm{idem}}$.

Remark 4.5.11. The name "partially additive blueprint" is derived from the fact that a partially additive blueprint B is characterized by its underlying monoid $B^\bullet$ and the partial functions $\Sigma_n : B^n \dashrightarrow B$ (for $n \geqslant 1$) that are defined as follows: the domain of Σ_n consists of all $(a_1, \dots, a_n) \in B^n$ such that $\sum a_i \in B$, and the value of such an element is $\Sigma_n(a_1, \dots, a_n) = \sum a_i$.

This notion is closely connected to Deitmar's theory of sesquiads in [Dei13]. Namely, a sesquiad corresponds naturally to a partially additive and cancellative blueprint; cf. Remark 2.9 in [Lor15] for more details.

Compatibility with quotients

Some of the properties considered above are compatible with taking quotients. We will explain some of such compatibilities in the following lemma.

Lemma 4.5.12. *Let B be a blueprint and $\mathfrak{c}$ be a congruence on B. Assume that $B /\!\!/ \mathfrak{c}$ is nontrivial. If B is a semiring, a blue field, with inverses or idempotent, then $B /\!\!/ \mathfrak{c}$ is so, too.*

Proof. We proof the claim case by case. Let $B = (R^\bullet, R)$ be a semiring. Then $R^\bullet / \mathfrak{c}^\bullet = (R/\mathfrak{c})^\bullet$ and $B /\!\!/ \mathfrak{c} = ((R/\mathfrak{c})^\bullet, R/\mathfrak{c})$ is a semiring.

Let B be a blue field, i.e. $B^\times = B - \{0\}$. Since $B /\!\!/ \mathfrak{c}$ is nontrivial, the quotient map $\pi : B \to B /\!\!/ \mathfrak{c}$ maps units $a \in B^\times$ to nonzero elements $\pi(a)$ of $B /\!\!/ \mathfrak{c}$. Thus π restricts to a surjection $B^\times \to (B /\!\!/ \mathfrak{c}) - \{0\}$. Thus every nonzero element of $B /\!\!/ \mathfrak{c}$ is of the form $\pi(a)$ for some $a \in B$, and $\pi(a^{-1})$ is a multiplicative inverse of $\pi(a)$. This shows that $B /\!\!/ \mathfrak{c}$ is a blue field.

Let B be with inverses, which is equivalent to the existence of a morphism $\mathbb{F}_{1^2} \to B$ by Lemma 4.5.2, part (5). Thus we gain a morphism $\mathbb{F}_{1^2} \to B \to B /\!\!/ \mathfrak{c}$, which shows that $B /\!\!/ \mathfrak{c}$ is with inverses.

Let B be idempotent, which is equivalent to the existence of a morphism $\mathbb{B} \to B$ by Lemma 4.5.2, part (6). Thus we gain a morphism $\mathbb{B} \to B \to B /\!\!/ \mathfrak{c}$, which shows that $B /\!\!/ \mathfrak{c}$ is idempotent. $\square$

4.6 Ideals

We have seen already that the notion of an ideal has different generalizations to semirings as congruences, ideals and k-ideals, depending on our purpose. The situation for monoids is similar. For blueprints, there are even more meaningful generalizations. We have already introduced congruences for blueprints. In this section, we define ideals, k-ideals and m-ideals and discuss their properties.

Definition 4.6.1. Let B be a blueprint. An *m-ideal* or *monoid ideal of B* is an ideal I of the monoid with zero $B^\bullet$. An *ideal of B* is an m-ideal I of B such that for all $a_1,\dots,a_n \in I$ and $b \in B$, an equality $b = \sum a_i$ in B^+ implies $b \in I$. A *k-ideal of B* is an m-ideal I of B such that for all $a_1,\dots,a_n,b_1\dots,b_m \in I$ and $c \in B$, an equality $\sum a_i + c = \sum b_j$ in B^+ implies $c \in I$.

It is apparent from the definition that every k-ideal is an ideal and that every ideal is an m-ideal. If $B \simeq A^{\text{blue}}$ for a monoid with zero A, then an m-ideal of B is the same as an ideal of A. If $B \simeq R^{\text{blue}}$ for a semiring R, then a (k-)ideal of B is the same as a (k)-ideal of R.

Lemma 4.6.2. *Let $f : B \to C$ be a blueprint morphism and I an (m/k-)ideal of C. Then $f^{-1}(I)$ is an (m/k-)ideal of B.*

Proof. Let I be an m-ideal of C and $J = f^{-1}(I)$. By definition, I is an ideal of $C^\bullet$. By Lemma 3.5.5, J is an ideal of $B^\bullet$, i.e. it is an m-ideal of B.

Let I be an ideal and consider $b = \sum a_i$ in B^+ with $b \in B$ and $a_i \in J$. Then $f(b) = \sum f(a_i)$ with $f(a_i) \in I$. Thus $f(b) \in I$ since I is an ideal, and $b \in J$. This shows that J is an ideal.

Let I be a k-ideal and consider $\sum a_i + c = \sum b_j$ in B^+ with $c \in B$ and $a_i, b_j \in J$. Then $\sum f(a_i) + f(c) = \sum f(b_j)$ with $f(a_i), f(b_j) \in I$. Thus $f(c) \in I$ since I is a k-ideal, and $c \in J$. This shows that J is a k-ideal. □

Lemma 4.6.3. *Let B be a blueprint and I a (k)-ideal of B^+. Then $I \cap B^\bullet$ is a (k)-ideal of B. If I is the (k)-ideal of B^+ generated by a (k)-ideal J of B, then $J = I \cap B^\bullet$.*

Proof. We begin with a general observation. Let I be a (k)-ideal of B^+. Then I is, in particular, an ideal of the multiplicative monoid of B^+ and $J = I \cap B^\bullet$ is the inverse image of I with respect to the inclusion $B^\bullet \to B^+$, which is a morphism of monoids with zero. Thus J is an m-ideal of B.

Let I be an ideal of B^+ and $J = I \cap B^\bullet$. Consider an equality $b = \sum a_i$ in B^+ with $a_i \in J$ and $b \in B$. Then $\sum a_i \in I$ and $b \in I$. Thus $b \in J = I \cap B^\bullet$, which shows that J is an ideal of B. This proves the first claim for ideals.

Let I be a k-ideal of B^+ and $J = I \cap B^\bullet$. Consider an equality $\sum a_i + c = \sum b_j$ in B^+ with $a_i, b_j \in J$ and $c \in B$. Then $a = \sum a_i$ and $b = \sum b_j$ are elements in the k-ideal I and thus $a + c = b$ implies that $c \in I$. Thus $c \in J = I \cap B^\bullet$, which shows that J is a (k)-ideal of B. This proves the first claim for k-ideals.

Let J be an ideal of B and $I = \langle J \rangle$ the ideal of B^+ generated by J. It is clear that $J \subset I \cap B$. By Corollary 2.5.4, we know that $I = \{\sum a_i s_i | a_i \in R, s_i \in J\}$. Since J is an ideal, we have in fact $I = \{\sum a_i | a_i \in J\}$. We conclude that if $b = \sum a_i$ is in $I \cap B^\bullet$, then $b \in J$ since J is an ideal. Thus $J = I \cap B^\bullet$ as claimed.

Let J be a (k)-ideal of B and $I = \langle J \rangle_k$ the (k)-ideal of B^+ generated by J. Clearly, $J \subset I \cap B$. By Corollary 2.5.4, we know that $I = \{c \in B^+ | a + c = b \text{ for some } a, b \in \langle J \rangle\}$. Let $c \in I \cap B^\bullet$. Then there are $a_i, b_j \in J$ such that $\sum a_i + c = \sum b_j$ by the characterization of I. Since J is a k-ideal of B, we conclude that $c \in J$ and thus $J = I \cap B^\bullet$ as claimed. This concludes the proof of the lemma. □

As a consequence, we derive in the following statement an explicit description for the smallest (m/k-)ideal of a blueprint B containing a given subset S. We call this (m/k-)ideal, the *(m/k-)ideal generated by S*.

Corollary 4.6.4. *Let B be a blueprint and S a subset of B. Then the m-ideal generated by S is*

$$\langle S \rangle_m \;=\; \left\{\, as \,\middle|\, a \in B, s \in S \cup \{0\} \,\right\},$$

the ideal generated by S is

$$\langle S\rangle \;=\; \left\{\, \sum a_i \,\middle|\, a_i \in \langle S\rangle_m \,\right\}$$

and the k-ideal generated by S is

$$\langle S\rangle_k \;=\; \left\{\, c\in B \,\middle|\, a+c=b \text{ in } B^+ \text{ for some } a,b\in\langle S\rangle \,\right\}.$$

Proof. The claim for m-ideals follows from the corresponding fact for monoids with zero, cf. section 3.4. The claim for ideals and k-ideals follows from combining Lemma 4.6.3 with Corollary 2.5.4. $\square$

Another consequence is the following.

Corollary 4.6.5. *Let B be a blueprint whose ambient semiring B^+ is a ring. Then every ideal of B is a k-ideal.*

Proof. Let I be an ideal of B and I^+ the ideal of B^+ generated by I. Then I^+ is a k-ideal of B^+ since B^+ is a ring and thus $I=I^+\cap B$ is a k-ideal of B by Lemma 4.6.3. $\square$

Exercise 4.6.6. Let B be a cancellative blueprint and I an ideal of B. Consider the ring $B^+_{\mathbb{Z}} = B^+\otimes^+_{\mathbb{N}}\mathbb{Z} = (B\otimes_{\mathbb{F}_1}\mathbb{F}_{1^2})^+$ as a blueprint and let $\iota : B\to B^+_{\mathbb{Z}}$ the morphism that sends a to $a\otimes 1$. Let $J=\langle\iota(I)\rangle$ be the ideal of $B^+_{\mathbb{Z}}$ that is generated by $\iota(I)$. Show that I is a k-ideal of B if and only if $I=\iota^{-1}(J)$. *Hint:* Use Exercise 2.5.8 and Lemma 4.6.3.

Exercise 4.6.7. Let B be a cancellative blueprint, I a k-ideal of B and $\mathfrak{c}=\mathfrak{c}(I)$ the congruence generated by I. Show that $B/\!\!/\mathfrak{c}$ is cancellative.

Definition 4.6.8. Let $f:B\to C$ be a morphism of blueprints. The *kernel of f* is the subset $\ker f = f^{-1}(0)$ of B.

Proposition 4.6.9. *The kernel of a blueprint morphism is a k-ideal and every k-ideal appears as the kernel of a blueprint morphism. More precisely, let B be a blueprint, I a k-ideal of B and $\mathfrak{c}=\mathfrak{c}(I)$ the congruence on B^+ generated by I. Then I is the kernel of the quotient morphism $\pi : B\to B/\!\!/\mathfrak{c}$.*

Proof. Let $f:B\to C$ be a blueprint morphism and $f^+:B\to C$ the morphism between the ambient semirings. By Proposition 2.5.3, $\ker f^+$ is a k-ideal of the semiring B^+ and by Lemma 4.6.3, $\ker f = \ker f^+\cap B$ is a k-ideal of B.

Let B be a blueprint, I a k-ideal of B and $I^+=\{\sum a_i | a_i\in I\}$ the ideal of B^+ generated by I. Then $\mathfrak{c}=\mathfrak{c}(I)$ is contained in $\mathfrak{c}(I^+)$. By Proposition 2.5.3, $a\sim_{\mathfrak{c}(I^+)} b$ implies $a+c=b+d$ for some $c,d\in I^+$, i.e. $c=\sum c_k$ and $d=\sum d_l$ for some $c_k,d_l\in I$. Since $c_k\sim_{\mathfrak{c}} 0\sim_{\mathfrak{c}} d_l$, we have

$$a \;\sim_{\mathfrak{c}}\; a+\sum c_k \;=\; b+\sum d_l \;\sim_{\mathfrak{c}}\; b,$$

which shows that $\mathfrak{c}=\mathfrak{c}(I^+)$.

Let $\pi : B\to B/\!\!/\mathfrak{c}$ be the quotient morphism. Using Proposition 2.5.3 once again, we see that $\ker\pi^+$ is the k-ideal of B^+ generated by I^+. Since I^+ is generated by I as an ideal, $\ker\pi^+$ is generated by I as a k-ideal. Thus by Lemma 4.6.3, $I=\ker\pi^+\cap B=\ker\pi$, as claimed. $\square$

We summarize the relations between the different notions of ideals and congruences for semirings, monoids with zero and blueprints in the following picture:

$$
\begin{tikzcd}
\{\text{ideals of } B^\bullet\} \arrow[rrr, shift right, hook] \arrow[d, leftrightarrow, "1:1"] & & & \{\text{congruences on } B^\bullet\} \arrow[lll, shift right, two heads] \arrow[d, shift right, hook] \\
\underset{\text{global}}{\{m\text{-ideals of } B\}} \arrow[r, shift right, two heads] & \{\text{ideals of } B\} \arrow[l, shift right, hook'] \arrow[r, shift right, two heads] \arrow[d, shift right, hook] & \{k\text{-ideals of } B\} \arrow[l, shift right, hook'] \arrow[r, shift right, hook] \arrow[d, shift right, hook] & \{\text{congruences on } B\} \arrow[l, shift right, two heads] \arrow[u, shift right, two heads] \arrow[d, leftrightarrow, "1:1"] \\
& \{\text{ideals of } B^+\} \arrow[r, shift right, two heads] \arrow[u, shift right, two heads] & \{k\text{-ideals of } B^+\} \arrow[l, shift right, hook'] \arrow[r, shift right, hook] \arrow[u, shift right, two heads] & \{\text{congruences on } B^+\} \arrow[l, shift right, two heads]
\end{tikzcd}
$$

Exercise 4.6.10. Conclude from the previous results that for every pair of an injection i and a surjection p between two sets in the above diagram, $p \circ i$ is the identity. Show that "paths in the same diagonal direction" commute, i.e. every subdiagram that does neither contain both an up arrow and a down arrow nor contain both a left arrow and a right arrow is commutative.

4.7 Prime ideals

Definition 4.7.1. Let B be a blueprint. An (m/k-)ideal I of B is *proper* if $I \neq B$. It is *prime* if $S = B - I$ is a multiplicative subset. An (m/k-)ideal I is *maximal* if it is proper and $I \subset J$ implies $I = J$ for every other proper (m/k-)ideal J of B.

Note that as in the case of semirings, an m-ideal I of B is proper or prime as an m-ideal if and only if it is proper or prime, respectively, as an ideal or a as k-ideal. Moreover, every k-ideal that is a maximal ideal is a maximal k-ideal and every ideal that is a maximal m-ideal is a maximal ideal. However, a maximal k-ideal does not need to be a maximal ideal, and a maximal ideal does not need to be a maximal m-ideal; cf. Example 4.7.8.

Note that similar to the case of monoids with zero, every blueprint B has a unique maximal m-ideal, which is $\mathfrak{m} = B - B^\times$. In this sense, every blueprint is local (with respect to m-ideals).

Lemma 4.7.2. *Let B be a blueprint. Then every maximal (m/k-)ideal of B is prime.*

Proof. The claim is immediate for m-ideals since $\mathfrak{m} = B - B^\times$ is the unique maximal m-ideal and the product of any element of B by a non-unit is a non-unit.

The proof for ideals and k-ideals is analogous to the case of semirings. We repeat the argument in brevity. Let $\mathfrak{m}$ be a maximal (k-)ideal and $ab \in \mathfrak{m}$ with $a \notin \mathfrak{m}$, i.e. B is generated by $S = \mathfrak{m} \cup \{a\}$ as a (k-)ideal. We want to show that $b \in \mathfrak{m}$.

In the case that $\mathfrak{m}$ is a maximal ideal, Corollary 4.6.4 implies that $1 = \sum e_k c_k$ for some $c_k \in S$ and $e_k \in B$. Since $bc_k \in \mathfrak{m}$, we obtain $b = \sum b e_k c_k \in \mathfrak{m}$, which shows that $\mathfrak{m}$ is prime.

In the case that $\mathfrak{m}$ is a maximal k-ideal, Corollary 4.6.4 implies that $\sum e_k c_k + 1 = \sum f_l d_l$ for some $c_k, d_l \in S$ and $e_k, f_k \in B$ and thus $\sum b e_k c_k + b = \sum b f_l d_l$. Since $b e_k c_k, b f_l d_l \in \mathfrak{m}$ and $\mathfrak{m}$ is a k-ideal, we conclude that $b \in \mathfrak{m}$ and that $\mathfrak{m}$ is prime. □

Exercise 4.7.3. Let B be a blueprint and I a proper (m/k-)ideal of B. Then I is contained in a maximal (m/k-)ideal of B. In particular, every nontrivial blueprint has a maximal (m/k-)ideal. *Hint:* The claims are obvious for m-ideals. For ideals and k-ideals, it follows from a standard application of the lemma of Zorn.

Lemma 4.7.4. *Let $f : B \to C$ be a morphism of blueprints and I a prime (m/k-)ideal of C. Then $f^{-1}(I)$ is a prime (m/k-)ideal of B.*

Proof. This follows from combining Lemmas 3.5.5 and 4.6.2. □

Exercise 4.7.5. Let B be a blueprint, I a k-ideal and $\mathfrak{c} = \mathfrak{c}(I)$ the congruence generated by I. Show that I is prime if and only if $B /\!\!/ \mathfrak{c}$ is without zero divisors. Find examples of a blueprint B and an ideal J of B for: (1) J is prime and $B /\!\!/ \mathfrak{c}(J)$ has zero divisors; (2) J is not prime and $B /\!\!/ \mathfrak{c}(J)$ is without zero divisors.

Lemma 4.7.6. *Let B be a blueprint and $S \subset B$ a subset that generates $B^\bullet$ over $B^\times$, i.e. for every $b \in B$, there are an element $a \in B^\times$ and elements $s_1, \dots, s_n \in S \cup \{0\}$ such that $b = as_1 \cdots s_n$. Then every prime m-ideal $\mathfrak{p}$ of B is generated by a subset J of S, i.e. $\mathfrak{p} = \langle J \rangle_m$.*

Proof. Let $J = S \cap \mathfrak{p}$. Then clearly $\langle J \rangle_m \subset \mathfrak{p}$. Consider $b \in B - \langle J \rangle_m$, i.e. $b = as_1, \dots, s_n$ for some $a \in B^\times$ and $s_1, \dots, s_n \in S - J$. By the definition of J and since $B^\times \cap \mathfrak{p} = \emptyset$, we have $a, s_1, \dots, s_n \in B - \mathfrak{p}$. Since $B - \mathfrak{p}$ is a multiplicative set, $b = as_1, \dots, s_n \in B - \mathfrak{p}$, which shows that $\mathfrak{p} = \langle J \rangle_m$ as claimed. □

Example 4.7.7. Let k be a blue field and $B = k[T_1, \dots, T_n]$. Then $B^\bullet$ is generated by $S = \{T_1, \dots, T_n\}$ over $B^\times$ and thus every prime m-ideal of B is generated by a subset of S. In this example, it is easily verified that for every subset J of S, the m-ideal $\mathfrak{p}_J = \langle J \rangle_m$ is prime and even a k-ideal.

More generally, if k is a blue field and $B = k[T_1, \dots, T_n] /\!\!/ \mathfrak{c}$, then we have inclusions

$$\{\text{prime } k\text{-ideals of } B\} \subset \{\text{prime ideals of } B\} \subset \{\text{prime } m\text{-ideals of } B\} \subset \{\text{subsets of } S\}$$

where $S = \{T_1, \dots, T_n\}$.

Example 4.7.8. The following example witnesses the digression between maximal m-ideals, maximal ideals and maximal k-ideals. Let $B = \mathbb{F}_1[T_1, T_2] /\!\!/ \langle T_1 + T_1 \equiv 1, T_2 + 1 \equiv T_2 \rangle$. By Lemma 4.7.2, every maximal (m/k-)ideal of B is prime. By Lemma 4.7.6, every prime m-ideal of B is generated by a subset of the generators T_1 and T_2 of B over the blue field $\mathbb{F}_1$. Thus it suffices to consider the (m/k-)ideals of B generated by the empty set, $\{T_1\}$, $\{T_2\}$ and $\{T_1, T_2\}$.

Since the unit group $B^\times = \{1\}$ does contain neither T_1 nor T_2, the unique maximal m-ideal of B is $B - B^\times = \langle T_1, T_2 \rangle_m$. If I is an ideal of B that contains T_1, then $1 = T_1 + T_1$ is also in I, i.e. $I = B$ is not proper. However, $\langle T_2 \rangle = T_2 \cdot B$ is a proper ideal since there is no relation of the form $\sum a_i = 1$ in B^+ with $a_i \in \langle T_2 \rangle$. Thus $\langle T_2 \rangle = \langle T_2 \rangle_m$ is the unique maximal ideal of B. If I is a k-ideal of B that contains T_2, then it also contains 1 since $T_2 + 1 = T_2$. Thus $\langle T_2 \rangle_k$ is not proper. We conclude that the only prime k-ideal is $\langle \emptyset \rangle_k = \{0\}$, which is henceforth the unique maximal k-ideal.

We see in this example that we have proper inclusions of $\{0\} \subsetneq \langle T_2 \rangle \subsetneq \langle T_1, T_2 \rangle_m$, and that the three different notions of maximality do not coincide.

4.8 Localizations

Definition 4.8.1. Let B be a blueprint and S a multiplicative subset of B. The *localization of B at S* is the blueprint $S^{-1}B = (S^{-1}B^\bullet, S^{-1}B^+)$. We write $B[h^{-1}] = S^{-1}B$ if $S = \{h^i\}_{i \in \mathbb{N}}$ for an element $h \in B$. We write $B_\mathfrak{p} = S^{-1}B$ if $S = B - \mathfrak{p}$ for a prime m-ideal $\mathfrak{p}$. If $S = B - \{0\}$ is a multiplicative subset of B, then we define the *fraction field of B* as $\operatorname{Frac} B = S^{-1}B$.

Note that $S^{-1}B$ is indeed a blueprint. First of all, S is clearly a multiplicative subset of B^+ with respect to the inclusion $B \hookrightarrow B^+$. Secondly, the induced map $S^{-1}B^\bullet \to S^{-1}B^+$ is injective since for elements $a, a' \in B$ and $s, s' \in S$, the fractions are equal in $S^{-1}B$ if and only if there is a $t \in S$ such that $tsa' = ts'a$, which, in turn, is equivalent to $\frac{a}{s} = \frac{a'}{s'}$ in $S^{-1}B^+$. This identifies $S^{-1}B^\bullet$ with a submonoid of $S^{-1}B^+$, which clearly contains the zero $\frac{0}{1}$ and the one $\frac{1}{1}$ and which generates $S^{-1}B^+$ as a semiring.

Note further that $(S^{-1}B)^\bullet = S^{-1}(B^\bullet)$ and $(S^{-1}B)^+ = S^{-1}(B^+)$ by the definition of the localization $S^{-1}B$. Therefore we can write $S^{-1}B^\bullet$ and $S^{-1}B^+$ without ambiguity. Finally note that the localization of B at S comes with the blueprint morphism $\iota_S : B \to S^{-1}B$ that sends a to $\frac{a}{1}$. This morphism satisfies $\iota_S(S) \subset (S^{-1}B)^\times$.

Example 4.8.2. We define $B[T^{\pm 1}]$ as $B[T][T^{-1}] = S^{-1}B[T]$ where $S = \{T^i\}_{i \in \mathbb{N}}$. If B is a blue field, then we have $B[T]_\mathfrak{p} = B[T]$ for $\mathfrak{p} = \langle T \rangle$ and $B[T]_\mathfrak{p} = \operatorname{Frac} B[T] = B[T^{\pm 1}]$.

Exercise 4.8.3. (Universal property of localizations) Let B be a blueprint, S a multiplicative subset and $\iota_S : B \to S^{-1}B$ the localization map. Show that for every blueprint morphism $f : B \to C$ such that $f(S) \subset C^\times$, there is a unique morphism $f_S : S^{-1}B \to C$ such that $f = f_S \circ \iota_S$.

Exercise 4.8.4. (Fraction fields) Let B be a blueprint and $S = B - \{0\}$. Show that S is a multiplicative set if and only if B is nontrivial and without zero divisors. In case that S is a multiplicative set, show that the localization map $B \to \operatorname{Frac} B$ is injective if and only if B is integral.

Localization is a very harmless operation on blueprints in the sense that it behaves well with basically all properties of blueprints that we have encountered in this chapter. Note that $S^{-1}B$ is trivial if $0 \in S$, which is why we exclude this case in the following statement.

Lemma 4.8.5. *Let B be a blueprint and S a multplicative subset of B that does not contain* 0*. If B is a monoid, a semiring, a blue field, integral, without zero divisors, with inverses, idempotent or cancellative, then $S^{-1}B$ is so, too.*

Proof. We prove the claim case by case. Let $B = (A, A^+)$ be a monoid. Then $S^{-1}(A^+) = (S^{-1}A)^+$ and thus $S^{-1}B = (S^{-1}A, (S^{-1}A)^+)$ is a monoid.

Let $B = (R^\bullet, R)$ be a semiring. Then $S^{-1}(R^\bullet) = (S^{-1}R)^\bullet$ and thus $S^{-1}R = ((S^{-1}R)^\bullet, S^{-1}R)$ is a semiring.

Let B be a blue field. Then $S^{-1}B = B$ is a blue field.

Let B be integral. By Exercise 4.8.4, $B \subset S^{-1}B \subset \operatorname{Frac} B$, which shows that $S^{-1}B$ is integral.

Let B without zero divisors and consider a product $\frac{a}{s} \cdot \frac{b}{t} = \frac{0}{1}$. Then there is a $w \in S$ such that $wab = wst \cdot 0 = 0$ in B. Since $w \neq 0$, we have $a =$ or $b = 0$. Thus $\frac{a}{s} = \frac{0}{1}$ or $\frac{b}{t} = \frac{0}{1}$, which shows that $S^{-1}B$ is without zero divisors.

Let B be with inverses, which is equivalent to the existence of a morphism $\mathbb{F}_{1^2} \to B$ by Lemma 4.5.2, part (5). Thus we gain a morphism $\mathbb{F}_{1^2} \to B \to S^{-1}B$, which shows that $S^{-1}B$ is with inverses.

Let B be idempotent, which is equivalent to the existence of a morphism $\mathbb{B} \to B$ by Lemma 4.5.2, part (6). Thus we gain a morphism $\mathbb{B} \to B \to S^{-1}B$, which shows that $S^{-1}B$ is idempotent.

Let B be cancellative and consider an equality $\frac{x}{s} + \frac{z}{v} = \frac{y}{t} + \frac{z}{v}$ in $S^{-1}B$. Then there is a $w \in S$ such that $wtvx + wstz = wsvy + wstz$ in B. Since B is cancellative, we have $wtvx = wsvy$ in B. Thus $\frac{x}{s} = \frac{y}{t}$ in $S^{-1}B$, which shows that $S^{-1}B$ is cancellative. □

Exercise 4.8.6. Let B be a partially additive blueprint, cf. Exercise 4.5.10, and S a multplicative subset of B. Show that $S^{-1}B$ is partially additive.

Lemma 4.8.7. *Let B be a blueprint, S a multiplicative subset and $\iota_S : B \to S^{-1}B$ the localization map. Let I be an (m/k-)ideal of B. Then*

$$S^{-1}I \;=\; \left\{ \tfrac{a}{s} \in S^{-1}B \,\middle|\, a \in I, s \in S \right\}$$

is the (m/k-)ideal of $S^{-1}B$ that is generated by $\iota_S(I)$.

Proof. Clearly we have $\iota_S(I) \subset S^{-1}I \subset \langle I \rangle_m$. Thus it suffices to show that $S^{-1}I$ is an (m/k-)ideal if I is so.

Let I be an m-ideal of B, which is the same as an ideal of the monoid $B^\bullet$. By Lemma 3.6.2, $S^{-1}I$ is an ideal of the monoid $S^{-1}B^\bullet$, which means that it is an m-ideal of $S^{-1}B$.

Let I be an ideal of B. We know already that $S^{-1}I$ is an m-ideal of $S^{-1}B$. Consider an equality $\frac{a}{s} = \sum \frac{a_i}{s_i}$ in $S^{-1}B^+$ with $a \in B$, $a_i \in I$ and $s, s_i \in S$. This means that there is a $t \in S$ such that

$$t\Big(\prod_{\text{all } i} s_i\Big)a \;=\; \sum_i ts\Big(\prod_{j\neq i} s_j\Big)a_i$$

holds in B^+. Since all terms $ts(\prod_{j\neq i} s_j)a_i$ are in I, also $t(\prod s_i)a$ is in I. Thus $\frac{a}{s} = \frac{t(\prod s_i)a}{t(\prod s_i)s}$ is in $S^{-1}I$. This shows that $S^{-1}I$ is an ideal of $S^{-1}B$.

Let I be a k-ideal of B. We know already that $S^{-1}I$ is an m-ideal of $S^{-1}B$. Consider an equality $\sum \frac{a_i}{s_i} + \frac{a}{s} = \sum \frac{a'_j}{s'_j}$ in $S^{-1}B^+$ with $a \in B$, $a_i, a'_j \in I$ and $s, s_i, s'_j \in S$. This means that there is a $t \in S$ such that

$$\sum_i ts\Big(\prod_{k\neq i} s_k\Big)\Big(\prod_{\text{all } j} s'_j\Big)a_i \;+\; t\Big(\prod_{\text{all } i} s_i\Big)\Big(\prod_{\text{all } j} s'_j\Big)a \;=\; \sum_j ts\Big(\prod_{\text{all } i} s_i\Big)\Big(\prod_{k\neq j} s'_k\Big)a'_j$$

holds in B^+. Since all terms $ts(\prod_{k\neq i} s_k)(\prod s'_j)a_i$ and $ts(\prod s_i)(\prod_{k\neq j} s'_k)a'_j$ are in I, we also have $t(\prod s_i)(\prod s'_j)a$ is in I. Thus $\frac{a}{s} = \frac{t(\prod s_i)(\prod s'_j)a}{t(\prod s_i)(\prod s'_j)s}$ is in $S^{-1}I$. This shows that $S^{-1}I$ is a k-ideal of $S^{-1}B$, which concludes the proof of the lemma. □

Proposition 4.8.8. *Let B be a blueprint, S a multiplicative subset of B and $\iota_S : B \to S^{-1}B$ the localization maps. Then the maps*

$$\begin{array}{ccc} \{\textit{prime m-ideals } \mathfrak{p} \textit{ of } B \textit{ with } \mathfrak{p}\cap S = \emptyset\} & \longleftrightarrow & \{\textit{prime m-ideals of } S^{-1}B\} \\ \mathfrak{p} & \overset{\Phi}{\longmapsto} & S^{-1}\mathfrak{p} \\ \iota_S^{-1}(\mathfrak{q}) & \overset{\Psi}{\longleftarrow\!\shortmid} & \mathfrak{q} \end{array}$$

are mutually inverse bijections. A prime m-ideal $\mathfrak{p}$ of B with $\mathfrak{p}\cap S = \emptyset$ is a (k-)ideal if and only if $S^{-1}\mathfrak{p}$ is a (k-)ideal.

Proof. The claim for m-ideals follows from Proposition 3.6.4. The claim for (k-)ideals follows from Lemmas 4.6.2 and 4.8.7. □

Residue fields

Let B be a blueprint, $\mathfrak{p}$ a prime m-ideal of B and $S = B - \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the complement of the units of $S^{-1}B$ and therefore its unique maximal m-ideal.

Definition 4.8.9. Let B be a blueprint and $\mathfrak{p}$ a prime m-ideal of B. The *residue field at* $\mathfrak{p}$ is the blueprint $k(\mathfrak{p}) = B_{\mathfrak{p}} /\!\!/ \mathfrak{c}(S^{-1}\mathfrak{p})$ where S is the complement of $\mathfrak{p}$ in B and $\mathfrak{c}(S^{-1}\mathfrak{p})$ is the congruence on $B_{\mathfrak{p}}^{+}$ that is generated by $S^{-1}\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime m-ideal of a blueprint B. Then the residue field at $\mathfrak{p}$ comes with a canonical morphism $B \to k(\mathfrak{p})$, which is the composition of the localization map $B \to B_{\mathfrak{p}}$ with the quotient map $B_{\mathfrak{p}} \to k(\mathfrak{p})$. Note that the residue field $k(\mathfrak{p})$ can be the trivial semiring in case that $\mathfrak{p}$ is not a k-ideal. More precisely, we have the following.

Corollary 4.8.10. *Let B be a blueprint, $\mathfrak{p}$ a prime m-ideal of B and $S = B - \mathfrak{p}$. Then the residue field $k(\mathfrak{p})$ is a blue field if $\mathfrak{p}$ is a k-ideal and trivial if not.*

Proof. First assume that $\mathfrak{p}$ is a prime k-ideal. Then $\mathfrak{p}$ is the maximal prime k-ideal that does not intersect S and thus $\mathfrak{m} = S^{-1}\mathfrak{p}$ is the unique maximal k-ideal of $S^{-1}B$ by Proposition 4.8.8, $\mathfrak{m}$ is a k-ideal. Thus the kernel of $S^{-1}B \to k(\mathfrak{p})$ is $\mathfrak{m}$, which shows that $k(\mathfrak{p})$ is not trivial. Since $(S^{-1}B)^{\times} = S^{-1}B - \mathfrak{m}$, we see that $(S^{-1}B)^{\times} \to k(\mathfrak{p}) - \{0\}$ is surjective, which shows that all nonzero elements of $k(\mathfrak{p})$ are invertible, i.e. $k(\mathfrak{p})$ is a blue field.

Next assume that $\mathfrak{p}$ is not a k-ideal. By Proposition 4.8.8, $\mathfrak{m} = S^{-1}\mathfrak{p}$ is not a k-ideal, which means that the kernel of $S^{-1}B \to k(\mathfrak{p})$ is strictly larger than $\mathfrak{m}$ and therefore contains a unit of $S^{-1}B$. This shows that $k(x)$ must be trivial. $\square$

Corollary 4.8.11. *Let B be a nontrivial blueprint. Then there exists a morphism $B \to k$ into a blue field k.*

Proof. By Exercise 4.7.3, B has a maximal k-ideal $\mathfrak{m}$. By Lemma 4.7.2, $\mathfrak{m}$ is prime. By Corollary 4.8.10, the residue field $k(\mathfrak{m})$ is a blue field, which provides a morphism $B \to k(\mathfrak{m})$ from B into a blue field $k(\mathfrak{m})$. $\square$

Corollary 4.8.12. *Let B be a blueprint and $\mathfrak{p}$ be a prime (k-)ideal of B. Then there is a prime (k-)ideal $\mathfrak{q}$ of B^{+} such that $\mathfrak{p} = \mathfrak{q} \cap B$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "\iota_{\mathfrak{p}}"] \arrow[d, "\alpha"] & B_{\mathfrak{p}} \arrow[d, "\alpha_{\mathfrak{p}}"] \\ B^{+} \arrow[r, "\iota_{\mathfrak{p}}^{+}"] & (B_{\mathfrak{p}})^{+} \end{tikzcd}$$

of blueprint morphisms. Let $S = B - \mathfrak{p}$. By Proposition 4.8.8, $S^{-1}\mathfrak{p}$ is the unique maximal (k-)ideal of $B_{\mathfrak{p}}$. Let I be the (k-)ideal of $(B_{\mathfrak{p}})^{+}$ generated by $\alpha_{\mathfrak{p}}(S^{-1}\mathfrak{p})$. By Lemma 4.6.3, we have $S^{-1}\mathfrak{p} = I \cap B_{\mathfrak{p}}$, which shows that I is a proper ideal of $B_{\mathfrak{p}})^{+}$. Exercise 4.7.3 shows that I is contained in a maximal (k-)ideal $\mathfrak{m}$ of $(B_{\mathfrak{p}})^{+}$, which is prime by Lemma 4.7.2. Thus $\alpha_{\mathfrak{p}}^{-1}(\mathfrak{m})$ is a prime (k-)ideal by Lemma 4.6.2 and thus $S^{-1}\mathfrak{p} \subset \alpha_{\mathfrak{p}}^{-1}(\mathfrak{m}) \subsetneq B_{\mathfrak{p}}$. By the maximality of $S^{-1}\mathfrak{p}$, we conclude that $S^{-1}\mathfrak{p} = \alpha_{\mathfrak{p}}^{-1}(\mathfrak{m})$.

Using Lemma 4.6.2 once again, we see that $\mathfrak{q} = (\iota_{\mathfrak{p}}^{+})^{-1}(\mathfrak{m})$ is a prime (k-)ideal of B^{+}. By the definition of $\mathfrak{q}$ and the commutativity of the diagram, we have that $\mathfrak{p} = \iota_{\mathfrak{p}}^{-1}\big(\alpha_{\mathfrak{p}}^{-1}(\mathfrak{m})\big) = \alpha^{-1}(\mathfrak{q})$, which concludes the proof of the corollary. $\square$

Exercise 4.8.13. Let B be a cancellative blueprint and $\mathfrak{p}$ a prime k-ideal of B. Show that there is a prime ideal $\mathfrak{q}$ of $B_{\mathbb{Z}}^{+}$ such that $\mathfrak{p} = \mathfrak{q} \cap B$. *Hint:* A slight alteration of the argument in the proof of Corollary 4.8.12, involving Lemmas 4.5.2 and 4.8.5 and Exercise 4.6.6, leads to success.

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Chapter 5

Ordered blueprints

In this chapter, we investigate ordered blueprints, which are an enhancement of blueprints by a partial order for the ambient semiring. Ordered blueprints were originally introduced in [Lor15].

5.1 Ordered semirings

To begin with, we recall the concept of a preorder and a partial order. Let S be a set. A *preorder on S* is a reflexive and transitive relation $\leqslant$ on S. A *partial order on S* is an antisymmetric preorder on S.

Definition 5.1.1. An *ordered semiring* is a semiring R together with a partial order $\leqslant$ on R that is additive and multiplicative, i.e. $x \leqslant y$ implies $x+z \leqslant y+z$ and $xz \leqslant yz$ for all $x,y,z \in R$. A *morphism of ordered semirings* $(R_1, \leqslant)$ *and* $(R_2, \leqslant)$ is a morphism $f : R_1 \to R_2$ of semirings that is order preserving, i.e. if $x \leqslant y$ in R_1, then $f(x) \leqslant f(y)$ in R_2. This defines the category OSRings of ordered semirings.

A *totally ordered semiring* is an ordered semiring $(R, \leqslant)$ whose partial order is total, i.e. either $a \leqslant b$ or $b \leqslant a$ for all $a,b \in R$. An ordered semiring $(R, \leqslant)$ is *algebraic* if the partial order $\leqslant$ is trivial, i.e. $x \leqslant y$ only if $x = y$. The *underlying semiring* or the *(algebraic) core of* $(R, \leqslant)$ is R. The *(algebraic) hull of* $(R, \leqslant)$ is the quotient semiring $R/\mathfrak{c}$ where $\mathfrak{c}$ is the congruence on R that is generated by $\{(a,b)|a \leqslant b\}$.

Remark 5.1.2. The terminologies "algebraic core" and "algebraic hull" stem from the corresponding usage for ordered blueprints where the addition of the semiring and the partial order can be bundled into one relation on the monoid semiring $\mathbb{N}[B^\bullet]$. We refer the reader to 5.6 for more details.

Example 5.1.3. We give a list of examples of ordered semirings. To begin with the trivial case: every semiring R together with the trivial order is an (algebraic) ordered semiring.

Specific examples are: the natural numbers $\mathbb{N}$ together with the natural total order; the nonnegative real numbers $\mathbb{R}_{\geqslant 0}$ together with the natural total order; the tropical numbers $\mathbb{T}$ together with the same total order as for $\mathbb{R}_{\geqslant 0}$; the Boolean numbers $\mathbb{B}$ together with the partial order generated by $0 \leqslant 1$, i.e. $a \leqslant b$ if and only if $a = 0$ or $b = 1$.

The last two examples are special cases of the following construction. Let G be an abelian group together with a total order $\leqslant$ such that $x \leqslant y$ implies $xz \leqslant yz$ for all $x,y,z \in G$ where we write G multiplicatively. Let $G_0 = G \cup \{0\}$, which is a monoid with zero 0 with respect to the extension of the multiplication of G to G_0 by $0 \cdot x = 0$ for all $x \in G_0$. We extend the partial order

$\leqslant$ of G to G_0 by $0 \leqslant x$ for all $x \in G_0$. Since $\leqslant$ is a total order on G_0, the maximum $\max\{x,y\}$ of any two elements $x,y \in G_0$ exists, and we define $x+y = \max\{x,y\}$. This addition turns G_0 into an idempotent semiring and $(G_0,\leqslant)$ into an ordered semiring.

Another specific example is the polynomial semiring $\mathbb{B}[T_1,\dots,T_n]$ with the following partial order. The support of a polynomial $f = \sum a_e T_1^{e_1} \cdots T_n^{e_n}$ is the set $\operatorname{supp} f$ of multi-indices $e = (e_1,\dots,e_n) \in \mathbb{N}^n$ for which $a_e \neq 0$. We define $f \leqslant g$ on $\mathbb{B}[T_1,\dots,T_n]$ if and only if $\operatorname{supp} f \subset \operatorname{supp} g$. This partial order turns $\mathbb{B}[T_1,\dots,T_n]$ into an ordered semiring.

Every semiring R comes with a natural additive and multiplicative preorder $\leqslant$, which we will define and study in Lemma 5.1.4 below. In many cases of interest, it turns out that $\leqslant$ is a partial order, i.e. $(R,\leqslant)$ is an ordered semiring. Note that all ordered semirings of Example 5.1.3 are of this type. Note further that the preorder $\leqslant$ also plays a central role in the definition of totally positive blueprints, cf. section 5.6.

Lemma 5.1.4. *Let R be a semiring and define $x \leqslant y$ if and only if $x+t = y$ for some $t \in R$. Then the following holds true.*

(1) The relation $\leqslant$ is the smallest additive and multiplicative preorder on R that contains $0 \leqslant 1$.

(2) The preorder $\leqslant$ is a partial order if and only if $x = x+s+t$ implies $x = x+s$ for all $x,s,t \in R$.

(3) If R is idempotent, then $\leqslant$ is a partial order. If $\leqslant$ is a partial order, then R is strict.

Proof. To begin with, we show that $\leqslant$ is an additive and multiplicative preorder that contains $0 \leqslant 1$. Since $x+0 = x$, we have $x \leqslant x$, i.e. $\leqslant$ is reflective. If $x \leqslant y$ and $y \leqslant z$, i.e. $x+s = y$ and $y+t = z$ for some $s,t \in R$. Then $x+s+t = y+t = z$ and $x \leqslant z$, i.e. $\leqslant$ is transitive. Let $x \leqslant y$, i.e. $x+t = y$ for some $t \in R$, and $z \in R$. Then $xz+tz = yz$ implies $xz \leqslant yz$ and $x+z+t = y+z$ implies $x+z \leqslant y+z$. Thus $\leqslant$ is additive and multiplicative. Since $0+1 = 1$, the preorder $\leqslant$ contains $0 \leqslant 1$.

Conversely, let $\leqslant'$ be an additive and multiplicative preorder containing $0 \leqslant' 1$. Consider an equation $x+t = y$ in R. Then we have

$$x \;=\; x+0\cdot t \;\leqslant'\; x+1\cdot t \;=\; y.$$

This shows that $\leqslant'$ contains $\leqslant$ and thus that $\leqslant$ is the smallest additive and multiplicative preorder containing $0 \leqslant 1$. This concludes the proof of (1).

The preorder $\leqslant$ is a partial order if and only if it is antisymmetric, i.e. $x \leqslant y$ and $y \leqslant x$ imply $x = y$. The inequalities $x \leqslant y$ and $y \leqslant x$ are, by the definition of $\leqslant$, equivalent to the existence of equalities $x+s = y$ and $x = y+t$ for some $s,t \in R$. Eliminating y by substituting y by $x+s$ yields the single equation $x = x+s+t$, and the required implication $x = y$ becomes $x = x+s$. Thus (2).

Let R be idempotent. Then $x+s+t = x$ implies

$$x+s \;=\; x+s+t+s \;=\; x+s+t \;=\; x.$$

By part (2), $\leqslant$ is a partial order, which proves the first claim of (3). Assume that $\leqslant$ is a partial order on R and $s+t = 0$ for some elements $s,t \in R$. Then $0+s+t = 0$ implies $s = 0+s = 0$ by (2), which shows that R is strict. This shows (3) and concludes the proof of the lemma. □

Exercise 5.1.5. Let R be an idempotent semiring and $\leqslant$ the partial order from Lemma 5.1.4. Show that $x \leqslant y$ if and only if $x+y = y$. Conclude that $\leqslant$ is a total order if and only if R is *bipotent*, i.e. $x+y \in \{x,y\}$ for all $x,y \in R$.

Exercise 5.1.6. Let R be a semiring and $\leqslant$ the preorder on R defined in Lemma 5.1.4. Show that the relation $\mathfrak{c} = \{(x,y) | x \leqslant y \text{ and } y \leqslant x\}$ is a congruence on R. Let $\pi : R \to R/\mathfrak{c}$ be the quotient map. Show that if $x \sim_{\mathfrak{c}} x'$ and $y \sim_{\mathfrak{c}} y'$, then $x \leqslant y$ if and only if $x' \leqslant y'$. Conclude that the rule $\pi(x) \leqslant \pi(y)$ whenever $x \leqslant y$ in R turns $R/\mathfrak{c}$ into an ordered semiring with $0 \leqslant 1$. Also confer Exercise 5.6.13.

Show that the quotient map $R \to R/\mathfrak{c}$ is a morphism $(R,=) \to (R/\mathfrak{c}, \leqslant)$ of ordered semirings and that it is universal for all morphisms from $(R,=)$ into ordered semirings $(S,\leqslant)$ with $0 \leqslant 1$.

Exercise 5.1.7. Let R be a semiring and $\mathfrak{c}$ the congruence defined in Exercise 5.1.6. Show that $R/\mathfrak{c}$ is trivial if and only if R is a ring. Show that, in fact, every ordered ring is algebraic.

As already observed in Example 5.1.3, every semiring R can be considered as an ordered semiring $(R,=)$ with respect to the trivial partial order. Evidently, a semiring morphism preserves the trivial partial orders. Thus we can and will think of the category SRings as a subcategory of OSRings. The embedding ι : SRings $\to$ OSRings is full since every morphism of ordered semirings is, by definition, a morphism of semirings.

Proposition 5.1.8. *The category* SRings *of semirings is a reflective and coreflective subcategory of the category* OSRings *of ordered semirings. The reflection* $(-)^{\text{hull}}$: OSRings $\to$ SRings *to the inclusion functor* ι : SRings $\to$ OSRings *sends an ordered semiring to its algebraic hull and the coreflection* $(-)^{\text{core}}$: OSRings $\to$ SRings *to* ι *sends an ordered semiring to its algebraic core.*

Proof. To begin with, we define the functors $(-)^{\text{core}}$ and $(-)^{\text{hull}}$. As described in the proposition, we put $(R,\leqslant)^{\text{core}} = R$ and $(R,\leqslant)^{\text{hull}} = R/\mathfrak{c}$ where $\mathfrak{c}$ is the congruence generated by $\{(x,y) | x \leqslant y\}$. Given an order preserving morphism $f : R \to S$ between ordered semirings $(R,\leqslant)$ and $(S,\leqslant)$, we define $f^{\text{core}} = f$, which defines $(-)^{\text{core}}$ for morphisms. The composition of f with the quotient map $\pi_S : S \to S^{\text{hull}}$ yields a semiring morphism $f' : R \to S^{\text{hull}}$. If $x \leqslant y$ in R, then $f(x) \leqslant f(y)$ in S and thus $f'(x) = \pi(f(x)) = \pi(f(y)) = f'(y)$ in S^{hull}. By the universal property of the quotient map $\pi_R : R \to R^{\text{hull}}$, cf. Lemma 2.4.8, there is a unique morphism $f^{\text{hull}} : R^{\text{hull}} \to S^{\text{hull}}$ such that $f' = f^{\text{hull}} \circ \pi_R$. This defines $(-)^{\text{hull}}$ for morphisms.

We proceed with showing that $(-)^{\text{core}}$ is right adjoint to the inclusion ι : SRings $\to$ OSRings. Let R be a semiring and $(S,\leqslant)$ an ordered semiring. Then every semiring morphism $f : R \to S$ is automatically order preserving since $x = y$ implies $f(x) = f(y)$ and thus $f(x) \leqslant f(y)$. This establishes a natural bijection $\operatorname{Hom}\big(R,(S,\leqslant)^{\text{core}}\big) \to \operatorname{Hom}\big((R,=),(S,\leqslant)\big)$, and shows that $(-)^{\text{core}}$ is right adjoint to ι.

We proceed with showing that $(-)^{\text{hull}}$ is left adjoint to the inclusion ι : SRings $\to$ OSRings. Let $(R,\leqslant)$ be an ordered semiring, S a semiring and $f : (R,\leqslant) \to (S,=)$ a morphism of ordered semirings. Using the identification $(S,=)^{\text{hull}} = S$, we obtain a morphism $f^{\text{hull}} : (R,\leqslant)^{\text{hull}} \to S$, which is the unique morphism with $f = f^{\text{hull}} \circ \pi_R$ where $\pi_R : R \to (R,\leqslant)^{\text{hull}}$ is the quotient map. This defines an injection $\operatorname{Hom}\big((R,\leqslant),(S,=)\big) \to \operatorname{Hom}\big((R,\leqslant)^{\text{hull}},S\big)$ whose surjectivity can be seen as follows. Given a morphism $g : (R,\leqslant)^{\text{hull}} \to S$, we obtain a semiring morphism $f = g \circ \pi_R : R \to S$. If $x \leqslant y$ in R, then $\pi_R(x) = \pi_R(y)$ and thus $f(x) = g \circ \pi_R(x) = g \circ \pi_R(y) = f(y)$. Therefore $f(x) \leqslant f(y)$ in $(S,=)$, which shows that f is order preserving. Thus $g = f^{\text{hull}}$, which completes the proof of the adjunction between ι and $(-)^{\text{hull}}$. $\square$

Corollary 5.1.9. *The embedding* ι : SRings $\to$ OSRings *commutes with both limits and colimits, i.e. we have* $\lim \iota(\mathcal{D}) \simeq \iota(\lim \mathcal{D})$ *and* $\operatorname{colim} \iota(\mathcal{D}) \simeq \iota(\operatorname{colim} \mathcal{D})$ *for every diagram* $\mathcal{D}$ *in* SRings. $\square$

Exercise 5.1.10. Show that OSRings is complete and cocomplete. In particular, show that $(\mathbb{N},=)$ is its initial object and $(\{0\},=)$ is its terminal object. Show that $(\lim \mathcal{D})^{\text{core}} \simeq \lim \mathcal{D}^{\text{core}}$ for every diagram $\mathcal{D}$ in OSRings, and that the partial order on $\lim \mathcal{D}$ is the richest partial order on $\lim \mathcal{D}^{\text{core}}$ such that all the canonical projections $\pi : \lim \mathcal{D}^{\text{core}} \to R$ to ordered semirings $(R,\leqslant)$ in $\mathcal{D}$ are order preserving.

Note that colimits do not commute with the the algebraic core, but require a quotient construction in general. For example, $\mathbb{Z} \otimes_{\mathbb{N}} \mathbb{N} = \mathbb{Z}$, but $(\mathbb{Z},=) \otimes_{(\mathbb{N},=)} (\mathbb{N},\leqslant) = (\{0\},=)$ if the partial order $\leqslant$ on $\mathbb{N}$ is nontrivial.

5.2 The definition of ordered blueprints

Definition 5.2.1. An *ordered blueprint* is a triple $B = (B^\bullet, B^+, \leqslant)$ such that $(B^\bullet, B^+)$ is a blueprint and $(B^+,\leqslant)$ is an ordered semiring. A *morphism $f : B \to C$ of ordered blueprints* is an order preserving semiring morphism $f^+ : B^+ \to C^+$ that maps $B^\bullet$ to $C^\bullet$. Let OBlpr be the category of ordered blueprints.

We call $B^\bullet$ the *underlying monoid*, B^+ the *ambient semiring* and $\leqslant$ the *partial order* of B.

Let $B = (B^\bullet, B^+, \leqslant)$ be an ordered blueprint. We think of $B^\bullet$ as the set of elements of B, i.e. we write $a \in B$ for $a \in B^\bullet$ and $S \subset B$ for $S \subset B^\bullet$. If we write $x \leqslant_B y$ or that $x \leqslant y$ holds in B, then we assume implicitly that x and y are elements B^+. Often we represent elements x of B^+ as sums $x = \sum a_i$ of elements a_i of $B^\bullet$. So we might write that $\sum a_i \leqslant \sum b_j$ holds in B where we assume implicitly that $a_i, b_j \in B^\bullet$. We write $\sum a_i \geqslant \sum b_j$ for $\sum b_j \leqslant \sum a_i$ and $\sum a_i \equiv \sum b_j$ for $\sum a_i \leqslant \sum b_j$ and $\sum a_i \geqslant \sum b_j$.

Note that a morphism $f : B \to C$ of ordered blueprints is determined by the restriction $f^\bullet : B^\bullet \to C^\bullet$, and that this restriction is a morphism of monoids with zero. A morphism $f : B \to C$ of ordered blueprints is *injective* if $f^\bullet : B^\bullet \to C^\bullet$ is injective, and it is *surjective* if $f^\bullet$ is surjective. Note that if $f : B \to C$ is surjective, then $f^+ : B^+ \to C^+$ is also surjective, but that the injectivity of f does not imply that f^+ is injective.

Several definitions for blueprints have a meaningful extension to ordered blueprints.

Definition 5.2.2. Let B be an ordered blueprint. The *unit group of B* is the unit group $B^\times$ of its underlying monoid $B^\bullet$. The *unit field of B* is the ordered blueprint $B^\star = B^\times \cup \{0\} /\!\!/ \mathfrak{r}$ where $\mathfrak{r} = \{\sum a_i \leqslant \sum b_j | \sum a_i \leqslant \sum b_j \text{ in } B\}$. An *ordered blue field* is an ordered blueprint B such that $B^\times = B^\bullet - \{0\}$.

The ordered blueprint B is *integral*, *without zero divisors*, *cancellative*, *idempotent* or *with -1* if the blueprint $(B^\bullet, B^+)$ is so.

Note that $B^\star$ is an ordered blue field unless B is trivial. We denote by $\text{OBlpr}^\star$ the full subcategory of OBlpr whose objects are ordered blue fields and the trivial ordered blueprint. We denote by $\text{OBlpr}^{\text{canc}}$ the full subcategory of cancellative ordered blueprints and by $\text{OBlpr}^{\text{idem}}$ the full subcategory of idempotent ordered blueprints.

Exercise 5.2.3. Formulate and prove a generalization of Lemma 4.5.2 to ordered blueprints.

Exercise 5.2.4. Show that $\text{OBlpr}^\star$ is a coreflective subcategory of OBlpr and that $\text{OBlpr}^{\text{canc}}$ and $\text{OBlpr}^{\text{idem}}$ are reflective subcategories of OBlpr.

First examples

Every ordered semiring $(R,\leqslant)$ can be considered as an ordered blueprint $(R^\bullet,R,\leqslant)$. We say that an ordered blueprint B is an *ordered semiring* if it is isomorphic to an ordered blueprint of the form $(R^\bullet,R\leqslant)$, i.e. if $B^\bullet=B^+$. The relation of OBlpr with Blpr and OSRings will be inspected in more detail in section 5.6.

Every blueprint $(B^\bullet,B^+)$ can be considered as an ordered blueprint $(B^\bullet,B^+,=)$ by endowing the ambient semiring B^+ with the trivial partial order. We say that an ordered blueprint is *algebraic* if it is isomorphic to an ordered blueprint of the form $(B^\bullet,B+,=)$, i.e. if the ambient semiring is trivially ordered. We briefly write *algebraic blueprint* for an algebraic ordered blueprint.

The ordered blueprint $\mathbb{F}_1=(\{0,1\},\mathbb{N},=)$ is the initial object in OBlpr, and the *trivial ordered blueprint* $(\{0\},\{0\},=)$ is a terminal object in OBlpr.

Other examples are $(\{0,1\},\mathbb{N},\leqslant)$ and $([0,1],\mathbb{R}_{\geqslant 0},\leqslant)$ where $\leqslant$ denotes the natural total order on $\mathbb{N}$ and $\mathbb{R}_{\geqslant 0}$, respectively. We refer the reader to section 5.6 for more examples.

Exercise 5.2.5. Classify all ordered blueprints B whose ambient semiring B^+ consists of two elements 0 and 1.

5.3 Quotients

Let $B=(B^\bullet,B^+,\leqslant)$ be an ordered blueprint and $\mathfrak{r}$ be a preorder on B^+. We write $x\leqslant_\mathfrak{r} y$ if $(x,y)\in\mathfrak{r}$. We write $x\equiv_\mathfrak{r} y$ if $x\leqslant y$ and $y\leqslant_\mathfrak{r} x$. The preorder $\mathfrak{r}$ is *additive* if $x\leqslant_\mathfrak{r} y$ implies $x+z\leqslant_\mathfrak{r} y+z$ for all $x,y,z\in B^+$ and it is *multiplicative* if $x\leqslant_\mathfrak{r} y$ implies $xz\leqslant_\mathfrak{r} yz$ for all $x,y,z\in B^+$.

Given a morphism $f:B\to C$ of ordered blueprints, we define the relation $\mathfrak{r}(f)=\{(x,y)\in B^+\times B^+|f(x)\leqslant_C f(y)\}$ on B^+.

Lemma 5.3.1. *Let $f:B\to C$ be a morphism of ordered blueprints. Then $\mathfrak{r}(f)$ is an additive and multiplicative preorder on B^+ that contains the partial order $\leqslant_B$ of B.*

Proof. The relation $\mathfrak{r}(f)$ is obviously reflective. If $f(x)\leqslant_C f(y)\leqslant_C f(z)$, then $f(x)\leqslant_C f(z)$, i.e. $\mathfrak{r}(f)$ is transitive and thus a preorder. If $f(x)\leqslant_C f(y)$, then $f(x+z)=f(x)+f(z)\leqslant f(y)+f(z)=f(y+z)$ and $f(xz)=f(x)f(z)\leqslant f(y)f(z)=f(yz)$, which verifies that $\mathfrak{r}(f)$ is additive and multiplicative. Since f is order preserving, $x\leqslant_B y$ in B implies $f(x)\leqslant_C f(y)$. This shows that $\mathfrak{r}(f)$ contains $\leqslant_B$, which completes the proof of the lemma. □

Given a subset S of $B^+\times B^+$, we denote by $\langle S\rangle$ the smallest additive and multiplicative preorder $\mathfrak{r}$ on B^+ that contains $\leqslant_B$ and S. Note that $\langle S\rangle$ is well-defined since the intersection of additive and multiplicative preorders is an additive and multiplicative preorder. Thus

$$\langle S\rangle \;=\; \bigcap \mathfrak{r}$$

where $\mathfrak{r}$ ranges over all additive and multiplicative preorders of B^+ that contain both $\leqslant_B$ and S.

Proposition 5.3.2. *Let B be an ordered blueprint and $\mathfrak{r}=\langle S\rangle$ the additive and multiplicative preorder on B^+ generated by a subset S a subset of $B^+\times B^+$. Then there exists an ordered blueprint $B/\!\!/\mathfrak{r}$ and a surjective morphism $\pi:B\to B/\!\!/\mathfrak{r}$ such that $\pi^+(x)\leqslant_{B/\!\!/\mathfrak{r}}\pi^+(y)$ if and only if $x\leqslant_\mathfrak{r} y$ for all $x,y\in B^+$. Every morphism $f:B\to C$ of ordered blueprints such that $f(x)\leqslant_C f(y)$ whenever $(x,y)\in S$ factors into $\overline{f}\circ\pi$ for a unique morphism $\overline{f}:B/\!\!/\mathfrak{r}\to C$.*

Proof. As a first step, we show that $\mathfrak{c} = \{(x,y) \in B^+ \times B^+ | x \equiv_{\mathfrak{r}} y\}$ is a congruence on B^+. It is evidently reflective and symmetric. To verify transitivity, consider $x \equiv_{\mathfrak{r}} y$ and $y \equiv_{\mathfrak{r}} z$, i.e. $x \leqslant_{\mathfrak{r}} y \leqslant_{\mathfrak{r}} z$ and $z \leqslant_{\mathfrak{r}} y \leqslant_{\mathfrak{r}} x$. By the transitivity of $\mathfrak{r}$, we conclude that $x \sim_{\mathfrak{c}} z$, which show that $\mathfrak{c}$ is transitive. To verify additivity and multiplicativity, consider $x \equiv_{\mathfrak{r}} y$, i.e. $x \leqslant_{\mathfrak{r}} y$ and $y \leqslant_{\mathfrak{r}} x$. Then we have $x+z \leqslant_{\mathfrak{r}} y+z$ and $y+z \leqslant_{\mathfrak{r}} x+z$ for every $z \in B^+$ and thus $x+z \equiv_{\mathfrak{r}} y+z$, which establishes the additivity of $\mathfrak{c}$. Similarly, we have $xz \leqslant_{\mathfrak{r}} yz$ and $yz \leqslant_{\mathfrak{r}} xz$ for every $z \in B^+$ and thus $xz \equiv_{\mathfrak{r}} yz$, which establishes the multiplicativity of $\mathfrak{c}$. To conclude, this verifies that $\mathfrak{c}$ is a congruence.

Let $\pi : B^+ \to B^+/\mathfrak{c}$ be the quotient map. As a second step, consider $x, x', y, y' \in B^+$ with $x \leqslant_{\mathfrak{r}} y$, $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$. Then we have $x' \leqslant_{\mathfrak{r}} x \leqslant_{\mathfrak{r}} y \leqslant_{\mathfrak{r}} y'$ by the definition of $\mathfrak{c}$, and thus $x' \leqslant_{\mathfrak{r}} y'$ by the transitivity of $\mathfrak{r}$. This shows that we can define a relation $\bar{\mathfrak{r}}$ on $\overline{B^+} = B^+/\mathfrak{c}$ by the rule $\pi(x) \leqslant_{\bar{\mathfrak{r}}} \pi(y)$ if and only if $x \leqslant_{\mathfrak{r}} y$.

It is easily seen that $\bar{\mathfrak{r}}$ inherits the properties of an additive and multiplicative preorder from $\mathfrak{r}$. If $\pi(x) \equiv_{B/\!\!/\mathfrak{r}} \pi(y)$, then $x \sim_{\mathfrak{c}} y$ and thus $\pi(x) = \pi(y)$ by the definition of $\mathfrak{c}$. Thus the preorder $\bar{\mathfrak{r}}$ is antisymmetric and turns $\overline{B^+}$ into an ordered semiring. Together with the multiplicative subset $\overline{B^\bullet} = \pi(B^\bullet)$, we obtain an ordered blueprint $B/\!\!/\mathfrak{r} = (\overline{B^\bullet}, \overline{B^+}, \bar{\mathfrak{r}})$.

It is evident that the quotient map π is a morphism of ordered blueprints $\pi : B \to B/\!\!/\mathfrak{r}$. By the definition of $\mathfrak{c}$, we have $\pi^+(x) \leqslant_{B/\!\!/\mathfrak{r}} \pi^+(y)$ if and only if $x \leqslant_{\mathfrak{r}} y$ as claimed.

We are left with the verification of the universal property of $\pi : B \to \overline{B}$. Let $f : B \to C$ be a morphism of ordered blueprints such that $f(x) \leqslant_C f(y)$ whenever $(x,y) \in S$. Since $\pi : B \to B/\!\!/\mathfrak{r}$ is surjective, a morphism $\overline{f} : B/\!\!/\mathfrak{r} \to C$ with $f = \overline{f} \circ \pi$ is necessarily uniquely defined by $\overline{f}(\pi(x)) = f(x)$. We are left with verifying the existence of $\overline{f}$.

By Lemma 5.3.1, the relation $\mathfrak{r}' = \{(x,y) \in B^+ \times B^+ | f(x) \leqslant_C f(y)\}$ is an additive and multiplicative preorder on B. It contains S by our assumptions on f and it contains $\leqslant_B$ since f is order preserving. Thus $\mathfrak{r}'$ contains $\mathfrak{r} = \langle S \rangle$. This means, first of all, that $f : B^+ \to C^+$ factors through $\pi : B^+ \to B^+/\mathfrak{c}$ as a semiring morphism by Proposition 2.4.4. It further implies that $\overline{f}$ is order preserving, i.e. a morphism of ordered semirings. Finally, it is evident that $\overline{f}(\overline{B^\bullet})$ is a subset of $C^\bullet$. This finishes the proof of the proposition. □

A *quotient* of an ordered blueprint B is an equivalence class of surjective morphisms $f : B \to C$ of ordered blueprints where two morphism $f : B \to C$ and $f' : B \to C'$ are equivalent if there is an isomorphism $g : C \to C'$ such that $f' = g \circ f$.

As a consequence of Lemma 5.3.1 and Proposition 5.3.2, we see that for an ordered blueprint B, the associations

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{additive and multiplicative} \\ \text{preorders on } B^+ \text{ containing } \leqslant_B \end{array} \right\} & \longleftrightarrow & \{\text{quotients of } B\} \\ \mathfrak{r} & \longmapsto & B \to B/\!\!/\mathfrak{r} \\ \mathfrak{r}(f) & \longleftarrow\!\shortmid & f : B \twoheadrightarrow C \end{array}$$

are mutually inverse bijections.

Notational conventions

Proposition 5.3.2 allows us to construct ordered blueprints from an algebraic blueprint B by endowing a set of relation on B^+, by which we mean a subset of $B^+ \times B^+$. Typically, we write such relations as $\sum a_i \leqslant \sum b_j$ where we implicitly assume that $a_i, b_j \in B$. Note that we can represent every element $x \in B^+$ as a sum $\sum a_i$ with $a_i \in B$. If $S = \{\sum a_{l,i} \leqslant \sum b_{l,j}\}_{l \in I}$ is a set of relations on B^+, then

$$B/\!\!/\langle S\rangle \;=\; B/\!\!/\langle \sum a_{l,i} \leqslant \sum b_{l,j}\rangle$$

is an ordered blueprint satisfying the relations in S. We write $\sum a_i \equiv \sum b_j$ for $\sum a_i \leqslant \sum b_j$ and $\sum b_j \leqslant \sum a_i$. Note that distinct elements $a,b \in B$ might be identified in $B/\!\!/\langle S\rangle$.

Example 5.3.3. This notation allows us to define ordered blueprints such as

$$\mathbb{B}^{\mathrm{pos}} \;=\; \mathbb{B}/\!\!/\langle 0 \leqslant 1\rangle \qquad \text{and} \qquad \mathbb{N}^{\mathrm{mon}} \;=\; \mathbb{N}^\bullet/\!\!/\langle a \leqslant b+c \,|\, a = b+c \text{ in } \mathbb{N}\rangle.$$

More explanations on $(-)^{\mathrm{pos}}$ and $(-)^{\mathrm{mon}}$ can be found in section 5.6.

5.4 Axiomatic ordered blueprints

The original definition of an ordered blueprint in [Lor15] differs from the approach taken in this text, but produces an equivalent notion of ordered blueprints. We will the definition of [Lor15] and explain the relationship between these two approaches in this section. To distinct the objects from [Lor15] from the ones in this text, we call them axiomatic ordered blueprints in the following.

Definition 5.4.1. An *axiomatic ordered blueprint* is a monoid with zero A together with a *subaddition on* A, which is a preorder $\mathcal{R}$ on the set $\mathbb{N}[A]^+ = \{\sum a_i | a_i \in A\}$ of finite formal sums of elements of A that satisfies for all $x,y,z,t \in \mathbb{N}[A]$ and $a,b \in A$ that

(1) $x \leqslant y$ and $z \leqslant t$ implies $x+z \leqslant y+t$ and $xz \leqslant yt$,

(2) $a \equiv b$ implies $a = b$ as elements of A, and

(3) $0_A \equiv 0_{\mathbb{N}[A]}$, i.e. the zero of A is equivalent to zero of $\mathbb{N}[A]$,

where we write $x \leqslant y$ for $(x,y) \in \mathcal{R}$ and $a \equiv b$ for $a \leqslant b$ and $b \leqslant a$. We also write $B^\bullet$ for A and say that $x \leqslant y$ holds in B if $(x,y) \in \mathcal{R}$. A *morphism between axiomatic ordered blueprints B and C* is a morphism $f : B^\bullet \to C^\bullet$ of monoids with zero such that for all $a_i, b_j \in B^\bullet$ with $\sum a_i \leqslant \sum b_j$ in B, we have $\sum f(a_i) \leqslant \sum f(b_j)$ in C. Let AxOBlpr be the category of axiomatic ordered blueprints.

From axiomatic ordered blueprints to ordered blueprints and back

Every ordered blueprint gives rise to an axiomatic ordered blueprint and, vice versa, an ordered blueprint can be recovered from its associated axiomatic ordered blueprint. In particular, note that both the information about the ambient semiring B^+ and the partial order $\leqslant$ on B^+ of an ordered blueprint B is bundled together in the subaddition $\mathcal{R}$.

In the following, we define two mutually inverse functors $(-)^{\mathrm{ax}} : \mathrm{OBlpr} \to \mathrm{AxOBlpr}$ and $(-)^{\mathrm{ob}} : \mathrm{AxOBlpr} \to \mathrm{OBlpr}$. We omit several details in the constructions of these functors and leave their verification as an exercise to the reader.

Let $B = (B^\bullet, B^+, \leqslant)$ be an ordered blueprint. We set

$$\mathcal{R}_B \;=\; \big\{ (\sum a_i, \sum b_j) \in \mathbb{N}[B^\bullet] \times \mathbb{N}[B^\bullet] \,\big|\, \sum a_i \leqslant \sum b_j \text{ in } (B^+, \leqslant) \big\}$$

and define the axiomatic ordered blueprint associated with B as $B^{\mathrm{ax}} = (B^\bullet, \mathcal{R}_B)$. Let $f : B \to C$ be a morphism of ordered blueprints. We define the associated morphism of axiomatic ordered blueprints as $f^{\mathrm{ax}} = f^\bullet : B^\bullet \to C^\bullet$. This finishes the construction of the functor $(-)^{\mathrm{ax}} : \mathrm{OBlpr} \to \mathrm{AxOBlpr}$.

We turn to the construction of $(-)^{\text{ob}} : \text{AxOBlpr} \to \text{OBlpr}$. Let $B = (A, \mathcal{R})$ be an axiomatic ordered blueprint. Since $0_A \equiv 0_{\mathbb{N}[A]}$, the subaddition $\mathcal{R}$ defines an additive and multiplicative preorder $\mathfrak{r}$ on $A^+ = \mathbb{N}[A]/\langle 0_A \equiv 0_{\mathbb{N}[A]}\rangle$. If we identify A with the ordered blueprint $(A, A^+, =)$, then $\mathfrak{r}$ contains clearly the trivial partial order $=$. By Proposition 5.3.2, we can form the quotient of A by $\mathfrak{r}$. We define $B^{\text{ob}} = A/\!\!/\mathfrak{r}$.

A morphism $f : B \to C$ of axiomatic ordered blueprints yields a morphism $f^\bullet : B^\bullet \to C^\bullet$ of monoids with zeros. Composing $f^\bullet$ with the quotient map $C^\bullet \to C^{\text{ob}}$ yields an ordered blueprint morphism $B^\bullet \to C^{\text{ob}}$, which factors through a unique ordered blueprint morphism $f^{\text{ob}} : B^{\text{ob}} \to C^{\text{ob}}$ since f preserves the subadditions of B and C. This finishes the construction of the functor $(-)^{\text{ob}}$.

Exercise 5.4.2. Show that the functors $(-)^{\text{ax}}$ and $(-)^{\text{ob}}$ are well-defined and that they are mutually inverse equivalences of categories between OBlpr and AxOBlpr.

Exercise 5.4.3. Let $B = (A, \mathcal{R})$ be an axiomatic ordered blueprint. Show that the relation $\mathfrak{c} = \{(\sum a_i, \sum b_j) | \sum a_i \equiv \sum b_j \text{ in } B\}$ is a congruence on the semiring $\mathbb{N}[A]$ and that $(B^{\text{ob}})^+$ is isomorphic to $\mathbb{N}[A]/\mathfrak{c}$.

5.5 Categorical constructions

In this section, we construct free ordered blueprints and tensor products, products and coproducts, equalizers and coequalizers. As a consequence, this shows that OBlpr is complete and cocomplete. Some details are left as an exercise.

Let B be an ordered blueprint and $S = \{T_i\}_{i \in I}$ be a set. We define the *free B-algebra in S* as the ordered blueprint

$$B[T_i]_{i\in I} \;=\; B[S] \;=\; B^\bullet[S]/\!\!/\langle \textstyle\sum a_i \leqslant \sum b_j \,|\, a_i, b_j \in B^\bullet \text{ and } \sum a_i \leqslant \sum b_j \text{ in } B\rangle$$

where we consider B as a subset of $B[S]$. The free algebra $B[S]$ comes together with an injective morphism $\iota_B : B \to B[S]$ of ordered blueprints and an injection $\iota_S : S \to B[S]$.

Exercise 5.5.1. Show that $(B[S])^\bullet = B^\bullet[S]$ and $(B[S])^+ = B^+[S]$. Show that the partial order of $B[S]$ is the smallest additive and multiplicative partial order on $B^+[S]$ that contains all relations $x \leqslant y$ that hold in $(B^+, \leqslant)$.

Lemma 5.5.2. *For every ordered blueprint morphism $f_B : B \to C$ and every map $f_S : S \to C$, there is a unique morphism $f : B[S] \to C$ of ordered blueprints such that $f_B = f \circ \iota_B$ and $f_S = f \circ \iota_S$.*

Proof. The morphism $f_B : B \to C$ yields a morphism $f^\bullet : B^\bullet \to C^\bullet$ of monoids with zero and $f_S : S \to C$ restricts to a map $f_S^\bullet : S \to C^\bullet$ by identifying C with its underlying set $C^\bullet$.

By Exercise 3.2.5, there is a unique morphism $f^\bullet : B^\bullet[S] \to C^\bullet$ of monoids such that $f_B^\bullet = f^\bullet \circ \iota_B^\bullet$ and $f_S^\bullet = f^\bullet \circ \iota_S^\bullet$.

We can identify the monoid with zero $C^\bullet$ with the algebraic blueprint $(C^\bullet, (C^\bullet)^+, =)$, which comes with a canonical morphism into C that extends the identity map $\iota_C : C^\bullet \to C$. Composing this morphism with $f^\bullet$ yields a morphism $\tilde{f} : B^\bullet[S] \to C$. Since $C^\bullet \to C$ is a bijection, $\tilde{f}$ is uniquely determined by $f^\bullet$.

Since $f : B \to C$ is a morphism, we have that if $x \leqslant y$ in B, then $\tilde{f}(x) \leqslant \tilde{f}(y)$ in C. By Proposition 5.3.2, the morphism $\tilde{f}$ factors uniquely through $B[S] = B^\bullet[S]/\!\!/\mathfrak{r}$ where $\mathfrak{r} = \{(x,y) \,|\, x \leqslant y \text{ in } B\}$. This completes the proof of the lemma. □

Let $f_C : B \to C$ and $f_D : B \to D$ be two ordered blueprint morphisms. We define the *tensor product of C and D over B* as

$$C \otimes_B D \;=\; C^\bullet \otimes_{B^\bullet} D^\bullet /\!\!/ \mathfrak{r}$$

where $\mathfrak{r}$ is the additive and multiplicative preorder on $(C^\bullet \otimes_{B^\bullet} D^\bullet)^+$ that is generated by all relations $\sum a_i \otimes 1 \leqslant \sum b_j \otimes 1$ for which $\sum a_i \leqslant \sum b_j$ in C and all relations $1 \otimes \sum c_k \leqslant 1 \otimes \sum d_l$ for which $\sum c_k \leqslant \sum d_l$ in D. The tensor product $C \otimes_B D$ comes together with the morphism $\iota_C : C \to C \otimes_B D$ that sends a to $a \otimes 1$ and the morphism $\iota_D : D \to C \otimes_C D$ that sends a to $1 \otimes a$.

Lemma 5.5.3. *The tensor product $C \otimes_B D$ is the colimit, or pushout, of $C \leftarrow B \to D$.*

Proof. Consider ordered blueprint morphisms $g_C : C \to E$ and $g_D : D \to E$ such that $g_C \circ f_C = g_D \circ f_D$. By Exercise 3.2.2, there is a unique morphism $g^\bullet : C^\bullet \otimes_{B^\bullet} D^\bullet \to E^\bullet$ such that $g_C^\bullet = g^\bullet \circ \iota_C^\bullet$ and $g_D^\bullet = g^\bullet \circ \iota_D^\bullet$. Composing $g^\bullet$ with the canonical morphism $E^\bullet \to E$ yields a morphism $\tilde{g} : C^\bullet \otimes_{B^\bullet} D^\bullet \to E$, which is uniquely determined by $g^\bullet$.

Given a generator $x \otimes 1 \leqslant y \otimes 1$ of $\mathfrak{r}$, coming from a relation $x \leqslant y$ in C, then $g_C^+(x) \leqslant g_C^+(y)$ and thus $\tilde{g}^+(x \otimes 1) \leqslant \tilde{g}^+(y \otimes 1)$. The same holds for a generator $1 \otimes x \leqslant 1 \otimes y$ coming from a relation $x \leqslant y$ in D. By Proposition 5.3.2, the morphism $\tilde{g}$ factors uniquely through the quotient $C \otimes_B D = C^\bullet \otimes_{B^\bullet} D^\bullet /\!\!/ \mathfrak{r}$.

This shows that there is a unique morphism $g : C \otimes_B D \to E$ with $g_C = g \circ \iota_C$ and $g_D = g \circ \iota_D$, which completes the proof of the lemma. □

Exercise 5.5.4. Show that $(C \otimes_B D)^+ = C^+ \otimes_{B^+}^+ D^+$. Give an example for which the canonical map $C^\bullet \otimes_{B^\bullet} D^\bullet \to (C \otimes_B D)^\bullet$ is not an isomorphism.

Exercise 5.5.5. Let B be an ordered blueprint and $\{T_i\}_{i\in I}$ a set. Show that $B[T_i] \simeq B \otimes_{\mathbb{F}_1} \mathbb{F}_1[T_i]$.

In the following, we construct products and coproducts, equalizer and coequalizer. We leave the verification of their universal properties to the reader, see Exercise 5.5.6.

Let $\{B_i\}_{i\in I}$ be a family of ordered blueprints. The product of the B_i is the ordered blueprint

$$\prod_{i\in I} B_i \;=\; \prod B_i^\bullet /\!\!/ \langle \Sigma(a_{i,k}) \leqslant \Sigma(b_{i,l}) \,|\, \sum a_{i,k} \leqslant \sum b_{i,l} \text{ in } B_i \text{ for all } i \in I \rangle,$$

together with the coordinate projections $\mathrm{pr}_j : \prod B_i \to B_j$.

Let $\iota_j^\bullet : B_j^\bullet \to \bigotimes B_i^\bullet$ be the canonical inclusion of $B_j^\bullet$ into the coproduct of the $B_i^\bullet$. The coproduct of the B_i is the ordered blueprint

$$\bigotimes_{i\in I} B_i \;=\; \bigotimes B_i^\bullet /\!\!/ \langle \sum \iota_j(a_k) \leqslant \sum \iota_j(b_l) \,|\, j \in I, a_k, b_j \in B_j^\bullet \text{ and } \sum a_k \leqslant \sum b_l \text{ in } B_j \rangle,$$

together with the coordinate inclusions $\iota_j : B_j \to \bigotimes B_i$ induced by $\iota_j^\bullet$.

Let $f : B \to C$ and $g : B \to C$ be two ordered blueprint morphisms. The equalizer of f and g is the ordered blueprint

$$\mathrm{eq}(f,g) \;=\; \mathrm{eq}(f^\bullet, g^\bullet) /\!\!/ \langle \sum a_i \leqslant \sum b_j \,|\, \sum a_i \leqslant \sum b_j \text{ in } B \rangle,$$

together with the inclusion $\mathrm{eq}(f,g) \to B$.

The coequalizer of f and g is the ordered blueprint

$$\mathrm{coeq}(f,g) \;=\; C /\!\!/ \langle f(a) \equiv g(a) \,|\, a \in B \rangle,$$

together with the quotient map $C \to \mathrm{coeq}(f,g)$.

Exercise 5.5.6. Verify the universal properties for the product, the coproduct, the equalizer and the coequalizer.

Exercise 5.5.7. Let $f : B \to C$ be a morphism of ordered blueprints. Show that f is a monomorphism if and only if both f^+ is injective. Show that f is an epimorphism if $f^\bullet$ is surjective. Show that if f is an isomorphism, then both $f^\bullet$ and f^+ are bijective. Give an example of an epimorphism that is not surjective and an example of a bijection that is not an isomorphism.

5.6 Reflective subcategories

In this section, we introduce and investigate various subcategories of OBlpr, which are of interest in later parts of these notes. Each of them is either reflective or coreflective.

Ordered semirings

As already observed before, every ordered semiring $(R, \leqslant)$ can be considered as an ordered blueprint $(R^\bullet, R, \leqslant)$. Since a morphism $f : (R, \leqslant) \to (S, \leqslant)$ maps $R^\bullet$ to $S^\bullet$, it is evidently a morphism between the associated ordered blueprint. This defines a functor $\iota : \mathrm{OSRings} \to \mathrm{OBlpr}$.

Lemma 5.6.1. *The functor* $\iota : \mathrm{OSRings} \to \mathrm{OBlpr}$ *has a left inverse and left adjoint* $\rho : \mathrm{OBlpr} \to \mathrm{OSRings}$ *that maps an ordered blueprint B to the ordered semiring* $(B^+, \leqslant)$*. Thus we can identify* OSRings *with a reflective subcategory of* OBlpr.

Proof. According to the claim of the lemma, we define $\rho(B) = (B^+, \leqslant_B)$ for an ordered blueprint B. An ordered blueprint morphism $f : B \to C$ is, in particular, an order preserving semiring morphism $\rho(f) : (B^+, \leqslant_B) \to (C^+, \leqslant_C)$. This defines $\rho : \mathrm{OBlpr} \to \mathrm{OSRings}$ as a functor.

If $(R, \leqslant)$ is an ordered semiring, then its image under ι is $B = (R^\bullet, R, \leqslant)$ and evidently $\rho(B) = (R, \leqslant)$ is naturally isomorphic to $(R, \leqslant)$. This shows that ρ is left inverse to ι and allows us to identify the category SRings with its essential image in OBlpr.

The ordered semiring $\rho(B)$, identified with the ordered blueprint $((B^+)^\bullet, B^+, \leqslant_B)$, comes together with a canonical morphism $\eta_B : B \to \rho(B)$, which is an isomorphism if B is in the essential image of ι, i.e. if $B^\bullet = B^+$. Composing a morphism $\rho(B) \to R$ with η_B yields a map

$$\Phi : \ \mathrm{Hom}(\rho(B), R) \ \longrightarrow \ \mathrm{Hom}(B, R).$$

It is easily verified that sending a morphism $f : B \to R$ to $\rho(f) : \rho(B) \to \rho(R) = R$ defines an inverse bijection to Φ. This shows that ρ is left adjoint to ι and completes the proof of the lemma. □

Algebraic blueprints

In this section, we inspect the properties of the association that sends a blueprint $(B^\bullet, B^+)$ to its associated ordered blueprint $(B^\bullet, B^+, =)$. First note that a blueprint morphism $f : B \to C$ is order preserving with respect to the trivial partial orders on B^+ and C^+. This defines a functor $\iota : \mathrm{Blpr} \to \mathrm{OBlpr}$.

Conversely, we can associate an algebraic blueprint associate with every ordered blueprint in different ways, which will give rise to a right adjoint and a left adjoint functor to ι.

Definition 5.6.2. Let B be an ordered blueprint. The *algebraic core of B* is the blueprint

$$B^{\text{core}} \;=\; B^{\bullet} /\!\!/ \langle \sum a_i \equiv \sum b_j \,|\, \sum a_i \equiv \sum b_j \text{ in } B \rangle.$$

The *algebraic hull of B* is the the blueprint

$$B^{\text{hull}} \;=\; B^{\bullet} /\!\!/ \langle \sum a_i \equiv \sum b_j \,|\, \sum a_i \leqslant \sum b_j \text{ in } B \rangle.$$

Note that $(B^{\text{core}})^{\bullet} = B^{\bullet}$ and $(B^{\text{core}})^{+} = B^{+}$. In other words, the algebraic core of an ordered blueprint B is $B^{\text{core}} = (B^{\bullet}, B^{+}, =)$ where we replace the partial order $\leqslant_B$ of B by the trivial partial order. The algebraic hull of B is equal to the quotient of the blueprint $(B^{\bullet}, B^{+})$ by the congruence $\mathfrak{c} = \langle x \equiv y | x \leqslant y \text{ in } B \rangle$.

If we consider B^{core} as an ordered blueprint, then the identity map $(B^{\text{core}})^{+} \to B^{+}$ defines a canonical morphism $B^{\text{core}} \to B$. If we consider the B^{hull} as an ordered blueprint, then the quotient map $B^{+} \to (B^{\text{hull}})^{+}$ defines a canonical morphism $B \to B^{\text{hull}}$.

Lemma 5.6.3. *Every ordered blueprint with -1 is an algebraic blueprint.*

Proof. By multiplication with -1, a relation $\sum a_i \leqslant \sum b_j$ implies that $\sum -a_i \leqslant \sum -b_j$ and thus

$$\sum b_j \;\equiv\; \sum b_j + \sum -a_i + \sum a_i \;\leqslant\; \sum b_j + \sum -b_j + \sum a_i \;\equiv\; \sum a_i,$$

which shows that, indeed, $\sum a_i \equiv \sum b_j$. □

Lemma 5.6.4. *The functor $\iota : \text{Blpr} \to \text{OBlpr}$ has a right adjoint and left inverse $(-)^{\text{core}} : \text{OBlpr} \to \text{Blpr}$ that sends an ordered blueprint B to its algebraic core B^{core} and it has a left adjoint and left inverse $(-)^{\text{hull}} : \text{OBlpr} \to \text{Blpr}$ that sends an ordered blueprint B to its algebraic hull B^{hull}. Thus we can identify* Blpr *with a reflective and coreflective subcategory of* OBlpr.

Proof. Every morphism $f : B \to C$ of ordered blueprints is tautologically a morphism between the algebraic cores $B^{\text{core}} = (B^{\bullet}, B^{+})$ and $C^{\text{core}} = (C^{\bullet}, C^{+})$. This defines $(-)^{\text{core}}$ as a functor.

Let B be an ordered blueprint and $\epsilon_B : B^{\text{core}} \to B$ be the morphism extending the identity map. If B is algebraic, then ϵ_B is an isomorphism, which shows that $(-)^{\text{core}}$ is left inverse to ι.

Let B be an algebraic blueprint and C an ordered blueprint. Composing a morphism $f : B \to C^{\text{core}}$ with $\epsilon_C : C^{\text{core}} \to C$ defines a map

$$\Phi : \; \text{Hom}(B, C^{\text{core}}) \;\longrightarrow\; \text{Hom}(B, C).$$

It is easily verified that sending a morphism $f : B \to C$ to $f^{\text{core}} : B = B^{\text{core}} \to C^{\text{core}}$ defines an inverse bijection to Φ. This shows that $(-)^{\text{core}}$ is right adjoint to ι.

We turn to the definition of $(-)^{\text{hull}}$ for morphisms. Let $f : B \to C$ be a morphism of ordered blueprints. Composing with the quotient map $C \to C^{\text{hull}}$ yields a morphism $\tilde{f} : B \to C^{\text{hull}}$. Let $\mathfrak{c} = \langle x \equiv y | x \leqslant y \text{ in } B \rangle$. Since f is order preserving and by the definition of C^{hull}, we have that $\tilde{f}(x) \equiv \tilde{f}(y)$ in C^{hull} whenever $x \equiv_{\mathfrak{c}} y$ in B. Applying Proposition 5.3.2 yields a morphism $f^{\text{hull}} : B^{\text{hull}} \to C^{\text{hull}}$, which is uniquely determined by the property that $\tilde{f}$ factors into the quotient map $B \to B^{\text{hull}}$ followed by f^{hull}. This defines $(-)^{\text{hull}}$ as a functor.

Let B be an ordered blueprint and $\eta_B : B \to B^{\text{hull}}$ be the quotient map. If B is algebraic, then η_B is an isomorphism, which shows that $(-)^{\text{hull}}$ is left inverse to ι.

Let B be an ordered blueprint and C an algebraic blueprint. Composing a morphism $f : B^{\text{hull}} \to C$ with $\eta_B : B^{\text{hull}} \to B$ defines a map

$$\Psi : \; \text{Hom}(B^{\text{hull}}, C) \;\longrightarrow\; \text{Hom}(B, C).$$

It is easily verified that sending a morphism $f : B \to C$ to $f^{\text{hull}} : B^{\text{hull}} \to C^{\text{hull}} = C$ defines an inverse bijection to Ψ. This shows that $(-)^{\text{hull}}$ is left adjoint to ι and completes the proof of the lemma. □

Monomial blueprints

Let B be an ordered blueprint. A *(left) monomial relation on B* is a relation of the form $a \leqslant \sum b_j$ in B where we assume that $a, b_j \in B^\bullet$, as usual. Monomial relations appear in the definition of a valuation, which motivates the following definition. We will apply this construction to extend the notion of valuations in Chapter 6.

Definition 5.6.5. Let B be an ordered blueprint. The *left monomial ordered blueprint* or, for short, the *monomial blueprint associated with B* is the ordered blueprint

$$B^{\mathrm{mon}} \ = \ B^\bullet /\!\!/ \langle a \leqslant \textstyle\sum b_j \,|\, a \leqslant \sum b_j \text{ holds in } B \rangle.$$

An ordered blueprint B is *monomial* if the canonical morphism $B^{\mathrm{mon}} \to B$ is an isomorphism. We define $\mathrm{OBlpr}^{\mathrm{mon}} \subset \mathrm{OBlpr}$ as the full subcategory of monomial blueprints.

Note that canonical morphism $B^{\mathrm{mon}} \to B$ is an isomorphism between the respective underlying monoids, but that $(B^{\mathrm{mon}})^+ = (B^\bullet)^+ \to B^+$ is in general not injective.

Lemma 5.6.6. *The category* $\mathrm{OBlpr}^{\mathrm{mon}}$ *is a coreflective subcategory of* OBlpr *whose coreflection* $(-)^{\mathrm{mon}} : \mathrm{OBlpr} \to \mathrm{OBlpr}^{\mathrm{mon}}$ *sends an ordered blueprint B to B^{mon}.*

Proof. Let $f : B \to C$ be a morphism of ordered blueprints. Then $f^\bullet : B^\bullet \to C^\bullet$ induces a blueprint morphism $\tilde{f} : (B^\bullet, (B^\bullet)^+) \to (C^\bullet, (C^\bullet)^+)$. Consider a generator $a \leqslant \sum b_j$ of the preorder of B^{mon}. Then this relation also holds in B and therefore $f(a) \leqslant \sum f(b_j)$ in C. By the definition of C^{mon}, this relation holds also in C^{mon}. This shows that $\tilde{f}$ defines a morphism $f^{\mathrm{mon}} : B^{\mathrm{mon}} \to C^{\mathrm{mon}}$ of ordered blueprints, which defines the functor $(-)^{\mathrm{mon}}$.

By definition, $\mathrm{OBlpr}^{\mathrm{mon}}$ is a full subcategory of OBlpr. Therefore we are left with showing that $(-)^{\mathrm{mon}}$ is right adjoint to the embedding of $\mathrm{OBlpr}^{\mathrm{mon}}$ into OBlpr as a subcategory.

Let B be a monomial blueprint and C an ordered blueprint. Composing a morphism $f : B \to C^{\mathrm{mon}}$ with the canonical morphism $C^{\mathrm{mon}} \to C$ yields a map

$$\Phi : \ \mathrm{Hom}(B, C^{\mathrm{mon}}) \ \longrightarrow \ \mathrm{Hom}(B, C).$$

Since $(C^{\mathrm{mon}})^\bullet = C^\bullet$ and $f^\bullet = \Phi(f)^\bullet$ as maps between the underlying monoids, it follows that an inverse bijection to Φ is given by sending a morphism $g : B \to C$ to $g^{\mathrm{mon}} : B = B^{\mathrm{mon}} \to C^{\mathrm{mon}}$. This show that $(-)^{\mathrm{mon}}$ is right adjoint to the embedding $\mathrm{OBlpr}^{\mathrm{mon}} \hookrightarrow \mathrm{OBlpr}$, which completes the proof of the lemma. $\square$

Example 5.6.7. Every monoid $B = (A, A^+, =)$ with zero is a monomial blueprint. Some other examples of monomial blueprints are $\mathbb{F}_1{}^{\mathrm{pos}} = \{0,1\} /\!\!/ \langle 0 \leqslant 1 \rangle$ and $\mathbb{F}_1^{\pm} = \{0,1,\epsilon\} /\!\!/ \langle 0 \leqslant 1 + \epsilon \rangle$ with $\epsilon^2 = 1$.

The most interesting class of examples for the purpose of tropicalizations are monomial blueprints associated with rings. Let $B = (R^\bullet, R, =)$ be a ring. Then the associated monomial blueprint is

$$B^{\mathrm{mon}} \ = \ B^\bullet /\!\!/ \langle c \leqslant a + b \,|\, c = a + b \text{ in } R \rangle$$

Totally positive blueprints

Definition 5.6.8. A *totally positive* blueprint is an ordered blueprint B that satisfies $0 \leqslant 1$. We denote the full subcategory of totally positive blueprints in OBlpr by $\mathrm{OBlpr}^{\mathrm{pos}}$.

Let B be an ordered blueprint. The *associated totally positive blueprint* is the totally positive blueprint

$$B^{\mathrm{pos}} \;=\; B/\!\!/\langle 0 \leqslant 1\rangle.$$

Note that the associated totally positive blueprint comes with a canonical morphism $B \to B^{\mathrm{pos}}$ and that B is totally positive if and only if $B \to B^{\mathrm{pos}}$ is an isomorphism. Note further that B^{pos} is isomorphic to the tensor product $B \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\mathrm{pos}}$ and that the canonical morphism $B \to B^{\mathrm{pos}}$ coincides with the canonical inclusion of $B \to B \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\mathrm{pos}}$.

Lemma 5.6.9. *Let B be an ordered blueprint. Then the following are equivalent.*

(1) B is totally positive.

(2) $0 \leqslant a$ for all $a \in B$.

(3) $\sum a_i + \sum c_k \leqslant \sum b_j$ implies $\sum a_i \leqslant \sum b_j$.

(4) There exists a morphism $\mathbb{F}_1^{\mathrm{pos}} \to B$.

The morphism $\mathbb{F}_1^{\mathrm{pos}} \to B$ is unique in case it exists.

Proof. Let B satisfy (1). Multiplying the relation $0 \leqslant 1$ by $a \in B$ yields (2).

Let B satisfy (2). A relation $\sum a_i + \sum c_k \leqslant \sum b_j$ implies $\sum a_i \equiv \sum a_i + \sum 0 \leqslant \sum a_i + \sum c_k \leqslant \sum b_j$, which is (3).

Let B satisfy (3). Then $0+1 \leqslant 1$ implies $0 \leqslant 1$ and the unique morphism $\mathbb{F}_1 \to B$ factors uniquely through $\mathbb{F}_1^{\mathrm{pos}} = \mathbb{F}_1/\!\!/\langle 0 \leqslant 1\rangle$. Thus (4).

Let B satisfy (4). Since $0 \leqslant 1$ in $\mathbb{F}_1^{\mathrm{pos}}$, the existence of a morphism $\mathbb{F}_1^{\mathrm{pos}} \to B$ implies that $0 \leqslant 1$ in B. Thus (1).

A morphism $\mathbb{F}_1^{\mathrm{pos}} \to B$ is unique since it is determined by the unique images of 0 and 1. □

Corollary 5.6.10. *Let B be an ordered blueprint.*

(1) If $a \leqslant 0$ in B, then $a = 0$ in B^{pos}.

(2) If $1 + \sum c_k \leqslant 0$ for some c_k in B, then B^{pos} is trivial. In particular, if B is with -1, then B^{pos} is trivial.

(3) The preorder $\langle 0 \leqslant 1\rangle$ on B is equal to

$$\mathfrak{r} \;=\; \Big\{ \big(\sum a_i, \sum b_j\big) \,\Big|\, \sum a_i + \sum c_k \leqslant_B \sum b_j \text{ for some } c_k \in B \Big\}.$$

(4) The quotient map $B \to B^{\mathrm{pos}}$ is an isomorphism between the respective underlying monoids if and only if

$$a + \sum c_k \leqslant b \quad \text{and} \quad b + \sum d_l \leqslant a \quad \text{imply} \quad a = b \quad \text{in} \quad B.$$

Proof. By Lemma 5.6.9 (2), we have $0 \leqslant a$ for all a in B^{pos}. If $a \leqslant 0$ in B, then $a \equiv 0$ in B^{pos}, which shows (1).

By Lemma 5.6.9 (3), a relation $1 + \sum c_k \leqslant 0$ in B implies $1 \leqslant 0$ in B^{pos}. Thus $0 \equiv 1$ by (1), which is equivalent with $B^{\mathrm{pos}} = 0$. This shows (2).

We turn to the proof of (3). It is easily verified that $\mathfrak{r}$ is an additive and multiplicative preorder on B^+. Inserting $c_k = 0$ in the definition of $\mathfrak{r}$ yields that $\leqslant_B$ is contained in $\mathfrak{r}$. Since $0+1 \leqslant_B 1$, we have $0 \leqslant 1$ in $\mathfrak{r}$. This shows that $\langle 0 \leqslant 1\rangle$ is contained in $\mathfrak{r}$.

Conversely, consider a relation $\sum a_i + \sum c_k \leqslant_B \sum b_j$ in B and let $\pi : B \to B^{\mathrm{pos}}$ be the quotient map. By Lemma 5.6.9, we have $\sum \pi(a_j) \leqslant \sum \pi(b_j)$ in B^{pos} and thus $\sum a_i \leqslant \sum b_j$ in $\langle 0 \leqslant 1 \rangle$. This completes the proof of (3).

We turn to the proof of (4). Assume that $\pi : B \to B^{\mathrm{pos}}$ is bijective and consider relations of the form $a + \sum c_k \leqslant b$ and $b + \sum d_l \leqslant a$ in B. By Lemma 5.6.9 (3), we conclude that $\pi(a) \leqslant \pi(b)$ and $\pi(b) \leqslant \pi(a)$ in B^{pos} and thus $\pi(a) = \pi(b)$ by the antisymmetry of $\leqslant$. Since π is a bijection, we conclude that $a = b$ in B.

To show the converse implication, first note that it is enough to show that π is injective since it is surjective as a quotient map and since every bijective monoid morphism is an isomorphism. Consider $a, b \in B$ such that $\pi(a) = \pi(b)$, i.e. $\pi(a) \leqslant \pi(b)$ and $\pi(b) \leqslant \pi(a)$. By (3), we have $a + \sum c_k \leqslant_B b$ and $b + \sum_B d_l \leqslant a$ for some c_k and d_l in B. Using the assumptions on B, we conclude that $a = b$, which shows that π is injective. This shows (4). □

Example 5.6.11. The examples of most importance for the text are the totally positive blueprints associated with the nonnegative real numbers $\mathbb{R}_{\geqslant 0}$, the tropical numbers $\mathbb{T}$ and the Boolean numbers $\mathbb{B}$. By Lemma 5.6.10, each of the quotient maps $R_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}^{\mathrm{pos}}$, $\mathbb{T} \to \mathbb{T}^{\mathrm{pos}}$ and $\mathbb{B} \to \mathbb{B}^{\mathrm{pos}}$ are bijections and the partial order on the associated totally positive blueprint is given by $x \leqslant y$ if and only if $x + t = y$ for some t. We conclude that in each case, we obtain the natural total order, which was already the theme of Lemma 5.1.4.

See Exercise 5.6.13 for the general relationship between the natural preorder of a semiring, as considered in Lemma 5.1.4, and the associated totally positive blueprint.

We have already seen that B^{pos} is canonically isomorphic to $B \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\mathrm{pos}}$. The identification $(-)^{\mathrm{pos}} = - \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\mathrm{pos}}$ extends $(-)^{\mathrm{pos}}$ to a functor. Since B^{pos} is totally positive, its image is contained in $\mathrm{OBlpr}^{\mathrm{pos}}$.

Lemma 5.6.12. *The category* $\mathrm{OBlpr}^{\mathrm{pos}}$ *is a reflective subcategory of* OBlpr *whose reflection is* $(-)^{\mathrm{pos}} : \mathrm{OBlpr} \to \mathrm{OBlpr}^{\mathrm{pos}}$.

Proof. By its definition, $\mathrm{OBlpr}^{\mathrm{pos}}$ is a full subcategory of OBlpr. Let B be an ordered blueprint and C a totally positive blueprint. The composition of a morphism $f : B^{\mathrm{pos}} \to C$ with the quotient map $B \to B^{\mathrm{pos}}$ yields a map

$$\Phi : \ \mathrm{Hom}(B^{\mathrm{pos}}, C) \longrightarrow \mathrm{Hom}(B, C).$$

Since the canonical morphism $C \to C^{\mathrm{pos}}$ is an isomorphism, Φ possesses an inverse, which sends a morphism $g : B \to C$ to $g^{\mathrm{pos}} : B^{\mathrm{pos}} \to C^{\mathrm{pos}} = C$. It is easily verified that this latter association is indeed an inverse to Φ. This shows that $(-)^{\mathrm{pos}}$ is left adjoint to the embedding of $\mathrm{OBlpr}^{\mathrm{pos}}$ in OBlpr as a subcategory, which completes the proof of the lemma. □

Exercise 5.6.13. Let R be a semiring and $B = (R^\bullet, R, =)$ the associated ordered blueprint. Show that the natural preorder on R is equal to the relation $\langle 0 \leqslant 1 \rangle$ on $B^+ = R$. Let $(R/\mathfrak{c}, \leqslant)$ be the ordered semiring considered in Exercise 5.1.6 and $B' = ((R/\mathfrak{c})^\bullet, R/\mathfrak{c}, \leqslant)$ be the associated ordered blueprint. Conclude that $B' \simeq B^{\mathrm{pos}}$.

Exercise 5.6.14. Let R be a semiring and $\leqslant$ be the preorder on R that is defined by $x \leqslant y$ if and only if $x + t = y$ for some $t \in R$; cf. Lemma 5.1.4. Use Corollary 5.6.10 to reprove that $\leqslant$ is a partial order if and only if $x = x + s + t$ implies $x = x + s$.

Strictly conic blueprints

In this section, we encounter the question under which conditions B can be recovered from B^{pos}.

Definition 5.6.15. A *strictly conic (ordered) blueprint* is an ordered blueprint B such that the relations $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ imply $\sum a_i = \sum b_j$ in B^+. We denote the full subcategory of strictly conic blueprints in OBlpr by $\mathrm{OBlpr}^{\mathrm{conic}}$.

Let B be an ordered blueprint. The *strictly conic (ordered) blueprint associated with* B is the ordered blueprint $B^{\mathrm{conic}} = B/\!\!/\mathfrak{r}$ where $\mathfrak{r}$ is generated by

$$\left\{ \sum a_i \equiv \sum b_j \,\middle|\, \sum a_i + \sum c_k \leqslant \sum b_j \text{ and } \sum b_j + \sum d_l \leqslant \sum a_i \right\}.$$

Lemma 5.6.16. *The category* $\mathrm{OBlpr}^{\mathrm{conic}}$ *is a reflective subcategory of* OBlpr *whose reflection sends* B *to* B^{conic}.

Proof. As a first step, we define $(-)^{\mathrm{conic}}$ for morphisms. Let $f : B \to C$ be an ordered blueprint morphism. The composition with the quotient map $C \to C^{\mathrm{conic}}$ yields a morphism $\tilde{f} : B \to C^{\mathrm{conic}}$. Given relations $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ in B yield relations $\sum f(a_i) + \sum f(c_k) \leqslant \sum f(b_j)$ and $\sum f(b_j) + \sum f(d_l) \leqslant \sum f(a_i)$ in C and thus $\sum f(a_i) = \sum f(b_j)$ in B^{conic}. By Proposition 5.3.2, $\tilde{f}$ factors into the quotient map $B \to B^{\mathrm{conic}}$, followed by a unique morphism $f^{\mathrm{conic}} : B^{\mathrm{conic}} \to C^{\mathrm{conic}}$ of strictly conic blueprints. This defines $(-)^{\mathrm{conic}}$ as a functor.

Since the inclusion functor $\iota : \mathrm{OBlpr}^{\mathrm{conic}} \to \mathrm{OBlpr}$ is full by definition, we are left with showing that $(-)^{\mathrm{conic}}$ is left adjoint to ι. Let B be an ordered blueprint and C a strictly conic blueprint. The composition with the quotient map $B \to B^{\mathrm{conic}}$ defines a map

$$\Phi : \ \mathrm{Hom}(B^{\mathrm{conic}}, C) \longrightarrow \mathrm{Hom}(B, C).$$

It is easily verified that $C = C^{\mathrm{conic}}$ and that the association that sends a morphism $f : B \to C$ to $f^{\mathrm{conic}} : B^{\mathrm{conic}} \to C^{\mathrm{conic}} = C$ is an inverse bijection of Φ. This shows that $(-)^{\mathrm{conic}}$ is left adjoint to ι and completes the proof of the lemma. □

In order to investigate the relation between an ordered blueprint B and $(B^{\mathrm{pos}})^{\mathrm{core}}$, consider the commutative diagram

$$\begin{tikzcd}
& B \arrow[dr, "\alpha_B^{\mathrm{pos}}"] & \\
B^{\mathrm{core}} \arrow[ur, "\alpha_B^{\mathrm{core}}"] \arrow[dr, "\beta_B = (\alpha_B^{\mathrm{pos}})^{\mathrm{core}}"'] & & B^{\mathrm{pos}} \\
& (B^{\mathrm{pos}})^{\mathrm{core}} \arrow[ur, "\alpha_{B^{\mathrm{pos}}}^{\mathrm{core}}"'] &
\end{tikzcd}$$

where $\alpha_C^{\mathrm{core}} : C^{\mathrm{core}} \to C$ denotes the canonical morphism induced by the identity map, with C standing for B and B^{pos}, and where $\alpha_B^{\mathrm{pos}} : B \to B^{\mathrm{pos}}$ is the quotient map.

Proposition 5.6.17. *The map* β_B *is an isomorphism of blueprints if and only if* B *is strictly conic.*

Proof. Assume that β_B is an isomorphism. Since $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ in B imply $\sum a_i \equiv \sum b_j$ in B^{pos}, and therefore in $(B^{\mathrm{pos}})^{\mathrm{core}}$, this must also hold in B^{core} as β_B is an isomorphism. By the definition of the algebraic core, $\sum a_i \equiv \sum b_j$ in B, which shows that B is strictly conic.

To prove the reverse direction, assume that B is strictly conic. By Corollary 5.6.10 (3), α_B^{pos} is an isomorphism between the underlying monoids. The maps α_B^{core} and $\alpha_{B^{\mathrm{pos}}}^{\mathrm{core}}$ are so, too, by the definition of the algebraic core. This shows that β_B is an isomorphism between the underlying monoids.

Given an equality $\sum a_i \equiv \sum b_j$ in $(B^{\mathrm{pos}})^{\mathrm{core}}$, this must already hold in B^{pos}. By the definition of B^{pos}, there must be relations of the form $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ in B. As B is strictly conic, we have $\sum a_i \equiv \sum b_j$ in B and therefore in B^{core}. This shows that β_B is an isomorphism. □

Corollary 5.6.18. *Let B be a strictly conic ordered blueprint. Then $(B^{\mathrm{pos}})^{\mathrm{core}} \otimes_{B^{\mathrm{core}}} B \simeq B$.*

Proof. By Proposition 5.6.17, $\beta_B : B^{\mathrm{core}} \to (B^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism and thus the claim. □

Exercise 5.6.19. Let $\mathrm{Blpr}^{\mathrm{conic}} \subset \mathrm{OBlpr}$ be the full subcategory of strictly conic algebraic blueprints and B an algebraic blueprint. Show that $(B^{\mathrm{pos}})^{\mathrm{core}}$ is isomorphic to B if and only if B is strictly conic. Conclude that $(-)^{\mathrm{pos}}$ embeds $\mathrm{Blpr}^{\mathrm{conic}}$ fully faithfully into $\mathrm{OBlpr}^{\mathrm{pos}}$, with left-inverse $(-)^{\mathrm{core}}$.

Corollary 5.6.20. *Let B be a semiring and $\leqslant$ the natural preorder that is defined by $x \leqslant y$ if and only if $x + t = y$ for some $t \in B$. Then the following are equivalent.*

(1) B is strictly conic.

(2) $\beta_B : B = B^{\mathrm{core}} \to (B^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism.

(3) $x + s + t = x$ implies $x + s = x$.

(4) The preorder $\leqslant$ is antisymmetric.

(5) $B \to B^{\mathrm{pos}}$ is an isomorphism between the underlying monoids.

Proof. The equivalence of (1) and (2) is Proposition 5.6.17. The equivalence of (3) and (4) is Lemma 5.1.4 (2). The equivalence of (3) and (5) is Corollary 5.6.10 (4).

The equivalence of (1) with (3) is a mere reformulation: since B is a semiring, all sums are contained in B and every inequality is an equality. Thus with $x = \sum a_i$, $y = \sum b_j$, $s = \sum c_k$, $t = \sum d_l$, the conditions $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_j \leqslant \sum a_i$ become $x + s = y$ and $y + t = x$. Eliminating y yields $x + s + t = x$. Similarly, $\sum a_i = \sum b_j$ is equivalent to $x = x + s$. Thus the equivalence of (1) and (3). □

Recall that a semiring is strict if $x + y = 0$ implies $x = 0$. A *nonnegative blueprint* is a blueprint B such that the only element $a \in B$ with $a \leqslant 0$ is $a = 0$.

Lemma 5.6.21. *The following holds true.*

(1) A strictly conic semiring is strict.

(2) An idempotent algebraic blueprint is strictly conic.

(3) A totally positive blueprint is strictly conic.

(4) A nonnegative monomial blueprint is strictly conic.

Proof. Let B be a strictly conic semiring. Since $x + y = 0$ implies $0 + x + y = 0$ and thus $x = 0 + x = 0$, B is a strict semiring. Thus (1).

Let B be an idempotent algebraic blueprint and assume that $\sum a_i + \sum c_k = \sum b_j$ and $\sum b_j + \sum d_l = \sum a_i$. Then

$$\begin{aligned} \sum a_i \; &= \; \sum a_i + \sum a_i \; = \; \sum a_i + \sum b_j + \sum d_l \; = \; \sum a_i + \sum b_j + \sum b_j + \sum b_j + \sum d_l \\ &= \; \sum a_i + \sum a_i + \sum b_j + \sum b_j + \sum c_k + \sum d_l \\ &= \; \sum a_i + \sum a_i + \sum a_i + \sum b_j + \sum c_k \; = \; \sum a_i + \sum b_j + \sum c_k \; = \; \sum b_j + \sum b_j \; = \; \sum b_j, \end{aligned}$$

which shows that B is strictly conic. Thus (2).

If B is totally positive, then the relations $\sum a_i + \sum c_k = \sum b_j$ and $\sum b_j + \sum d_l = \sum a_i$ imply $\sum a_i \leqslant \sum b_j$ and $\sum b_j \leqslant \sum a_i$. Thus $\sum a_i = \sum b_j$ as desired. This shows (3).

Let B be non-negative and monomial and consider $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ where we assume that a_i, b_j, c_k, d_l are non-zero. These relations are generated by left monomial relations of the form $a' \leqslant \sum b'_j$, which contain at least one nonzero term b'_j if a' is nonzero since B is nonnegative. Therefore $\#\{a_i, c_k\} \leqslant \#\{b_j\}$ and $\#\{b_j, d_l\} \leqslant \#\{a_i\}$, which is only possible if $\{c_k\} = \{d_l\} = \emptyset$. Consequently, $\sum a_i = \sum b_j$, which shows that B is strictly conic as claimed in (4). □

Corollary 5.6.22. *If B is an idempotent algebraic blueprint, then $\beta_B : B^{\mathrm{core}} \to (B^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism and the quotient map $B \to B^{\mathrm{pos}}$ is a bijection.*

Proof. By Lemma 5.6.21, B is strictly conic and by Proposition 5.6.17, $B^{\mathrm{core}} \to (B^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism. Consequently, we obtain a bijection $B^{\mathrm{core}} = (B^{\mathrm{pos}})^{\mathrm{core}} \to B^{\mathrm{pos}}$, which factors into the canonical morphisms $B^{\mathrm{core}} \to B$ and $B \to B^{\mathrm{pos}}$. Since $B^{\mathrm{core}} \to B$ is a bijection, we conclude that $B \to B^{\mathrm{pos}}$ is also a bijection. □

Example 5.6.23 (A strict semiring that is not strictly conic)**.** The semiring $R = \mathbb{N}[S,T]^+ /\!\!/ \langle 1 + S + T \equiv 1 \rangle$ is obviously a strict semiring. However, $1 + S + T = 1$ while $1 + S \neq 1$, which shows that R is not strictly conic.

Exercise 5.6.24. Consider the following addition $+^t$ for the multiplicative monoid $\mathbb{R}^{\bullet}_{\geqslant 0}$:

$$a +^t b \quad = \quad \begin{cases} (a^t + b^t)^{1/t} & \text{if } t \in [1, \infty) \\ \max\{a, b\} & \text{if } t = \infty. \end{cases}$$

Show that $\mathbb{R}^{\bullet}_{\geqslant 0}$ together with the addition $+^t$ is a strictly conic semiring $\mathbb{R}^t_{\geqslant 0}$. Conclude that the quotient map $\mathbb{R}^t_{\geqslant 0} \to (\mathbb{R}^t_{\geqslant 0})^{\mathrm{pos}}$ is a bijection for all $t \in [1, \infty]$. Show that the partial order for $(\mathbb{R}^t_{\geqslant 0})^{\mathrm{pos}}$ is the natural total order for the nonnegative reals.

Note that $\mathbb{R}^1_{\geqslant 0} = \mathbb{R}_{\geqslant 0}$ and that $\mathbb{R}^{\infty}_{\geqslant 0} = \mathbb{T}$. Show that the identity map $\mathbb{T}^{\bullet} \to \mathbb{R}^{\bullet}_{\geqslant 0}$ induces a morphism $(\mathbb{T}^{\mathrm{pos}})^{\mathrm{mon}} \to (\mathbb{R}^{\mathrm{pos}}_{\geqslant 0})^{\mathrm{mon}}$ of ordered blueprints.

Ordered blueprints with unique weak inverses

Definition 5.6.25. Let B be an ordered blueprint and $a \in B$. A *weak inverse of a* is an element $b \in B$ such that $0 \leqslant a + b$. An ordered blueprint B is with *unique weak inverses* if every $a \in B$ has a unique weak inverse $b \in B$. We denote the weak inverse of 1 by ϵ. We denote by $\mathrm{OBlpr}^{\pm}$ the full subcategory of OBlpr whose objects are ordered blueprints with unique weak inverses.

Lemma 5.6.26. *Let B be an ordered blueprint with unique weak inverses and ϵ the weak inverse of 1. Then the following holds true.*

(1) *The weak inverse of $a \in B$ is ϵa.*

(2) *If a is a weak inverse of b, then b is a weak inverse of a.*

(3) $\epsilon^2 = 1$.

Proof. Multiplying the relation $0 \leqslant 1+\epsilon$ with a yields $0 \leqslant a+\epsilon a$; thus (1). Obviously, $0 \leqslant a+b$ implies $0 \leqslant b+a$; thus (2). It follows from (1) and (2) that both 1 and ϵ^2 are weak inverses of ϵ. By uniqueness of the weak inverse, we have (3). □

Example 5.6.27. An initial object of $\text{OBlpr}^{\pm}$ is

$$\mathbb{F}_1^{\pm} = \{0,1,\epsilon\} /\!\!/ \langle 0 \leqslant 1+\epsilon \rangle$$

where $\epsilon^2 = 1$. Another example is $\mathbb{F}_1 /\!\!/ \langle 0 \leqslant 1+1 \rangle$. Every blueprint with -1 is a blueprint with unique weak inverses.

Exercise 5.6.28. Let B be a strict semiring. Show that $B /\!\!/ \langle 0 \leqslant 1+1 \rangle$ is with unique weak inverses.

Exercise 5.6.29. Let B be an ordered blueprint. Show that B is algebraic and with unique weak inverses if and only if it is with -1.

The inclusion functor $\text{OBlpr}^{\pm} \to \text{OBlpr}$ turns out to have a left adjoint and left inverse $(-)^{\pm} : \text{OBlpr} \to \text{OBlpr}^{\pm}$, which can be described as follows.

Definition 5.6.30. Let B be an ordered blueprint. The *associated ordered blueprint with unique weak inverses* is

$$B^{\pm} \quad = \quad B \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\pm} /\!\!/ \langle a \equiv a' \,|\, 0 \leqslant a+b \text{ and } 0 \leqslant a'+b \text{ for some } b \in B \otimes_{\mathbb{F}_1} \mathbb{F}_1{}^{\pm} \rangle.$$

Exercise 5.6.31. Let B be an ordered blueprint. Show that $B^{\pm}$ is with unique weak inverses. Show that $\text{OBlpr}^{\pm}$ is a reflective subcategory of OBlpr whose reflection $(-)^{\pm} : \text{OBlpr} \to \text{OBlpr}^{\pm}$ sends an ordered blueprint to its associated ordered blueprint with unique weak inverses.

Reversible ordered blueprints

There is an interesting subclass of ordered blueprints with unique weak inverses that satisfy a reversibility for monomial relations. This class is of interest for its connections to hyperrings, cf. section 5.7.

Definition 5.6.32. A *reversible ordered blueprint* is an ordered blueprint B that contains an element ϵ with $\epsilon^2 = 1$ such that every relation $b \leqslant c + \sum a_i$ in B implies $\epsilon c \leqslant \epsilon b + \sum a_i$. We denote by $\text{OBlpr}^{\text{rev}} \subset \text{OBlpr}$ the full subcategory of reversible ordered blueprints.

Example 5.6.33. The ordered blueprint $\mathbb{F}_1^{\pm}$ is reversible with and an initial object in $\text{OBlpr}^{\text{rev}}$. Every blueprint with -1 is reversible with $\epsilon = -1$.

The ordered blueprint $\{0,1\} /\!\!/ \langle 0 \leqslant 1+1, 0 \leqslant 1+1+1 \rangle$ is with unique weak inverses since $0 \leqslant 1+\epsilon$ for $\epsilon = 1$, but not reversible since $0 \leqslant 1+1+1$, but not $\epsilon \leqslant 1+1$.

Lemma 5.6.34. *Every reversible ordered blueprint is with unique weak inverses. In particular, ϵ is the weak inverse of 1 and uniquely determined.*

Proof. Let B be a reversible ordered blueprint and $a \in B$. Then $a \leqslant 0+a$ implies $0 = \epsilon \cdot 0 \leqslant \epsilon a + a$, thus ϵa is a weak inverse of a. Assume that b is another weak inverse of a, i.e. $0 \leqslant a+b$. Then reversibility yields $\epsilon b \leqslant a$ and $\epsilon a \leqslant b$. Multiplication of the former relation with ϵ yields $b \leqslant \epsilon a$ and thus $b = \epsilon a$. This shows that ϵa is the unique weak inverse of a.

In particular ϵ is the weak inverse of 1 and therefore uniquely determined. This completes the proof of the lemma. □

Definition 5.6.35. Let B be an ordered blueprint. We define the associated *reversible ordered blueprint* as

$$B^{\mathrm{rev}} \;=\; B \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\pm} /\!\!/ \langle \epsilon c \leqslant \epsilon b + \textstyle\sum a_i \,\big|\, b \leqslant c + \sum a_i \text{ holds in } B \otimes_{\mathbb{F}_1} \mathbb{F}_1^{\pm} \rangle.$$

Exercise 5.6.36. Let B be an ordered blueprint. Show that B^{rev} is reversible. Show that $\mathrm{OBlpr}^{\mathrm{rev}}$ is a reflective subcategory of OBlpr whose reflection sends an ordered blueprint to its associated reversible ordered blueprint.

Exercise 5.6.37. Show that a reversible ordered blueprint is nonnegative, i.e. $a \leqslant 0$ implies $a = 0$. Show that a reversible and monomial ordered blueprint is strictly conic.

Exercise 5.6.38. Let B be an ordered blueprint with unique weak inverses whose partial order is generated by relations of the form $c \leqslant a+b$ with $a, b, c \in B^{\bullet}$. Show that B is reversible.

Overview of subcategories

Using the previous results on the relations between the different subcategories of OBlpr, we can illustrate some subcategories of OBlpr that are relevant to this text as in Figure 5.1. An inclusion of areas indicates an inclusion of subcategories, and areas with empty intersection correspond to subcategories that have only the trivial ordered blueprint in common. To be more precise, for every intersection of areas in the diagram, there is a nontrivial ordered blueprint that is contained in precisely those subcategories whose areas contain the intersection.

Note that the category HypRings of hyperrings and its embedding into OBlpr will be discussed in the following section 5.7.

Exercise 5.6.39. Show that the subcategories whose areas in Figure 5.1 have empty intersection have the trivial ordered blueprint as their only common object. Exhibit for every area of the diagram in Figure 5.1 an ordered blueprint with exactly the corresponding properties. For instance, find an ordered blueprint that is strictly conic with unique weak inverses, but that is neither reversible nor monomial.

Exercise 5.6.40. Determine, which areas in Figure 5.1 contain a nontrivial cancellative ordered blueprint. Insert an area in the diagram that represents the subcategory $\mathrm{OBlpr}^{\mathrm{canc}}$.

5.7 Relations to other theories

In this section, we investigate the relation of ordered blueprints to halos and hyperfields.

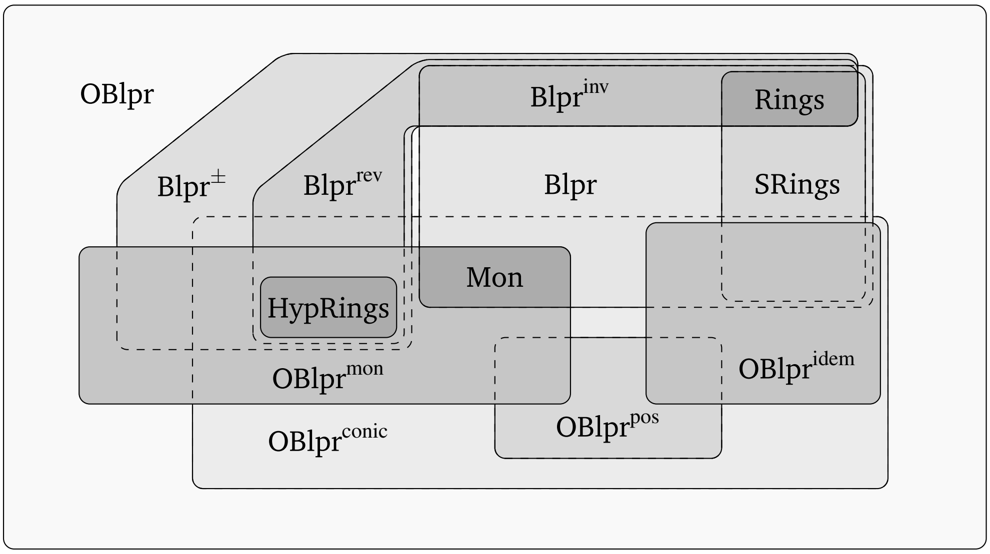

Figure 5.1: Some relevant subcategories of OBlpr

Halos

Paugam introduces in [Pau09] the notion of a halo, which allows us to consider absolute values of fields as morphisms in a category. This was a guiding idea in the development of ordered blueprints. Confer the introduction of [Lor15] for more details. It turns out that there exists a fully faithful functor from Halos to $\mathrm{OBlpr}^{\mathrm{mon}}$.

A *halo* is an ordered semiring and a *(multiplicative) halo morphism* is an order preserving multiplicative map $f : B_1 \to B_2$ of ordered semirings such that $f(0) = 0$, $f(1) = 1$ and $f(a+b) \leqslant f(a) + f(b)$. If we consider B_1 and B_2 as ordered semirings, then it is easily seen that a map $f : B_1 \to B_2$ is a halo morphism if and only if the composition $f' : B_1^{\mathrm{mon}} \to B_1 \to B_2$ is a morphism of ordered blueprints.

By Lemma 5.6.6, f' factors uniquely through the morphism $f^{\mathrm{mon}} : B_1^{\mathrm{mon}} \to B_2^{\mathrm{mon}}$ of monomial blueprints. This defines a functor $(-)^{\mathrm{mon}} : \mathrm{Halos} \to \mathrm{OBlpr}^{\mathrm{mon}}$ that sends a halo $(R, \leqslant)$ to B^{mon} where $B = (R^\bullet, R, \leqslant)$ is the ordered blueprint associated with $(R, \leqslant)$ as a semiring.

Exercise 5.7.1. Show that the functor $(-)^{\mathrm{mon}} : \mathrm{Halos} \to \mathrm{OBlpr}^{\mathrm{mon}}$ is fully faithful.

At the time of writing, it is not clear to the author if the category of halos contains tensor products. Since the obvious attempt to extend the partial orders to the tensor product of the underlying semirings fails to produce an object that satisfies the universal property of the tensor product, it seems likely that Halos does not contain all tensor products. A rigorous proof of this fact would be desirable.

As a consequence of the previous discussion, it seems unlikely that the embedding $(-)^{\mathrm{mon}} :$ $\mathrm{Halos} \to \mathrm{OBlpr}^{\mathrm{mon}}$ has a right or left adjoint, which would imply the existence of colimits since in this case, Halos would be equivalent to a reflective or coreflective subcategory of OBlpr.

Hyperrings

Another concept that has been tied to tropical geometry in recent years is that of a hyperring. Hyperrings first appeared in Krasner's paper [Kra57], and its connections to tropical geometry were studied by Viro in [Vir11].

Definition 5.7.2. A *commutative hypergroup* is a set G together with a distinctive element 0 and a *hyperaddition*, which is a map

$$\boxplus : \quad G \times G \quad \longrightarrow \quad \mathcal{P}(G)$$

into the power set $\mathcal{P}(G)$ of G, such that for all $a,b,c \in G$,

- $a \boxplus b$ is not empty, *(nonempty sums)*
- $\{a \boxplus d | d \in b \boxplus c\} = \{d \boxplus c | d \in a \boxplus b\}$, *(associativity)*
- $0 \boxplus a = a \boxplus 0 = \{a\}$, *(neutral element)*
- there is a unique element $-a$ in G such that $0 \in a \boxplus (-a)$, *(inverses)*
- $a \boxplus b = b \boxplus a$, *(commutativity)*
- $c \in a \boxplus b$ if and only if $(-a) \in (-c) \boxplus b$. *(reversibility)*

Note that thanks to the commutativity and associativity, it makes sense to define hypersums of several elements $a_1, \dots, a_n$ unambiguously by the recursive formula

$$\boxplus_{i=1}^{n} a_i = \left\{ b \boxplus a_n \,\middle|\, b \in \boxplus_{i=1}^{n-1} a_i \right\}.$$

Definition 5.7.3. A *(commutative) hyperring* is a set R together with distinctive elements 0 and 1 and with maps $\boxplus : R \times R \to \mathcal{P}(R)$ and $\cdot : R \times R \to R$ such that

- $(R, \boxplus, 0)$ is a commutative hypergroup,
- $(R, \cdot, 1)$ is a commutative monoid,
- $0 \cdot a = a \cdot 0 = 0$, and
- $a \cdot (b \boxplus c) = ab \boxplus ac$

for all $a,b,c \in R$ where $a \cdot (b \boxplus c) = \{ad | d \in b \boxplus c\}$. A *morphism of hyperrings* is a map $f : R_1 \to R_2$ between hyperrings such that

$$f(0) = 0, \qquad f(1) = 1, \qquad f(a \boxplus b) \subset f(a) \boxplus f(b) \qquad \text{and} \qquad f(ab) = f(a) \cdot f(b)$$

for all $a,b \in R_1$ where $f(a \boxplus b) = \{f(c) | c \in a \boxplus b\}$. We denote the category of hyperrings by HypRings.

We denote the underlying monoid of a hyperring R by $R^\bullet$. The unit group $R^\times$ of a hyperring R is the group of all multiplicatively invertible elements in R. A *hyperfield* is a hyperring K such that $K^\times = K - \{0\}$.

Example 5.7.4. Every ring R can be considered as a hyperring by defining $a \boxplus b = \{a+b\}$. If R is a field, the corresponding hyperring is a hyperfield.

The *Krasner hyperfield* $\mathbb{K} = \{0,1\}$ whose addition is characterized by $1 \boxplus 1 = \{0,1\}$. Note that all other sums and products are determined by the hyperring axioms.

The *tropical hyperfield* $\mathbb{T}^{\text{hyp}}$ was introduced by Viro in [Vir11]. Its multiplicative monoid are the non-negative real number $\mathbb{R}_{\geqslant 0}$ together with the usual multiplication and its hyperaddition is defined by the rule $a \boxplus b = \max\{a,b\}$ if $a \neq b$ and $a \boxplus a = [0,a]$.

The *sign hyperfield* $\mathbb{S}$ is the multiplicative monoid $\mathbb{S} = \{0, \pm 1\}$ and together with the hyperaddition characterized by $1 \boxplus 1 = \{1\}$, $(-1) \boxplus (-1) = \{-1\}$ and $1 \boxplus (-1) = \{-1, 0, 1\}$. Note that with this definition, the sign map $\mathbb{R} \to \mathbb{S}$ becomes a morphism of hyperfields.

All of the previous examples of hyperfields are instances of the following general construction of hyperfields as quotients of fields by a multiplicative subgroup. Let K be a field and G a multiplicative subgroup of $K^\times$. Then the quotient K/G of K by the action of G on K by multiplication carries a natural structure of a hyperfield: we have $(K/G)^\times = K^\times/G$ as an abelian group and

$$[a] \boxplus [b] \;=\; \left\{ \, [c] \;\middle|\; c = a' + b' \text{ for some } a' \in [a], b' \in [b] \, \right\}$$

for classes $[a]$ and $[b]$ of K/G.

The category of hyperrings embeds into OBlpr in the following way. Given a hyperring R, we define its associated ordered blueprint as

$$R^{\mathrm{ob}} \;=\; R^\bullet /\!\!/ \langle a \leqslant \textstyle\sum b_j \,|\, a \in \boxplus\, b_j \rangle.$$

Exercise 5.7.5. Show that every hyperring morphism $f : R_1 \to R_2$ defines a morphism $f^{\mathrm{ob}} : R_1^{\mathrm{ob}} \to R_2^{\mathrm{ob}}$ that maps $a \in R_1^{\mathrm{ob}}$ to $f(a) \in R_2^{\mathrm{ob}}$. Show that this defines a fully faithful functor $(-)^{\mathrm{ob}} : \mathrm{HypRings} \to \mathrm{OBlpr}$. Show that R^{ob} is monomial and reversible for every hyperring R.

Exercise 5.7.6. Show that the reversibility axiom for the hyperaddition follows from the other axioms of a hyperring.

Exercise 5.7.7. Let k be a field and $v : k \to \mathbb{R}_{\geqslant 0}$ a map. Show that v is a nonarchimedean absolute value if and only if it is a morphism $v : K \to \mathbb{T}^{\mathrm{hyp}}$ of hyperfields where we identify $\mathbb{R}_{\geqslant 0}$ and $\mathbb{T}^{\mathrm{hyp}}$ as sets.

Exercise 5.7.8. Let R be a bipotent semiring, i.e. $x + y \in \{x, y\}$ for all $x, y \in R$; also cf. Exercise 5.1.5. Define a hyperaddition $\boxplus$ by the rule $x \boxplus y = \{x + y\}$ if $x \neq y$ and $x \boxplus x = \{z \,|\, x + z = x\}$. Show that $R^\bullet$ together with $\boxplus$ forms a hyperring, which we denote by R^{hyp}. Show that the notation $\mathbb{T}^{\mathrm{hyp}}$ for the tropical hyperfield is compatible with this construction and that the Krasner hyperfield $\mathbb{K}$ is equal to $\mathbb{B}^{\mathrm{hyp}}$.

Remark 5.7.9. Jun develops a scheme theory for hyperrings in [Jun18], which fits naturally several aspects of tropical scheme theory. However, there are two drawbacks in this approach. One is that it is not clear if tensor products for hyperrings exist. In so far, we do not know if we can form fibre products of hyperring schemes. The other reason lies in the usage of prime hyperring ideals as the underlying points of hyperring schemes. This leads to the problem that the hyperring of global sections of the spectrum of a hyperring R is in general not equal to R.

Our approach of embedding hyperrings into ordered blueprints provides an alternative framework for hyperring schemes that circumvents both problems.

5.8 Ideals

Definition 5.8.1. Let B be an ordered blueprint and $\alpha : B^{\mathrm{core}} \to B$ be the identity map. A *(proper / prime / maximal) (m/k-)ideal of B* is a subset I of B such that $\alpha^{-1}(I)$ is a (proper / prime / maximal) (*m*/*k*-)ideal of B^{core}.

Thus the (m/k-)ideals of B are, by definition, the same as the (m/k-)ideals of B^{core}. Therefore all facts for the different types of ideals for algebraic blueprints carry over to ordered blueprints. For completeness, we list some important facts without proof.

Lemma 5.8.2. *Let $f : B \to C$ be an ordered blueprint morphism and I a (m/k-)ideal of C. Then $f^{-1}(I)$ is an (m/k-)ideal of B. If I is prime, then $f^{-1}(I)$ is also prime.* $\square$

Lemma 5.8.3. *Let B be an ordered blueprint. Then every maximal (m/k-)ideal of B is prime.* $\square$

Monomial ideals

Monomial ideals for ordered blueprints have been defined in [Dud17] as an example of a general theory that studies possible notions of ideals for algebraic structures. Monomial ideals can be seen as a generalizes of hyperring ideals to the context of ordered blueprints. See section 5.7 for details on hyperrings and [Jun18] for the definition of a hyperring ideal.

Definition 5.8.4. Let B be an ordered blueprint. A *monomial ideal* is an m-ideal of B such that $a \leqslant \sum_B b_j$ with $b_j \in B$ implies that $a \in B$. A monomial ideal I of B is *prime* if $S = B - I$ is a multiplicative subset of B, it is *proper* if $J \neq B$ and it is *maximal* if it is proper and if for every other proper monomial ideal J of B, an inclusion $I \subset J$ implies that $I = J$.

Exercise 5.8.5. Let $f : B \to C$ be a morphism of ordered blueprints and I a monomial ideal of C. Show that $f^{-1}(I)$ is a monomial ideal of B. Show that every maximal monomial ideal of B is prime. Show that every proper monomial ideal of B is contained in a maximal monomial ideal.

Give an example of an ordered blueprint B, an k-ideal I and a monomial ideal J of B such that I is not a monomial ideal and such that J is not an ideal of B.

5.9 Localizations

In this section, we define and investigate localizations of ordered blueprints. As a first step, we verify that the partial order of an ordered blueprint extends to a partial order of the localization of the ambient semiring.

Lemma 5.9.1. *Let $(R, \leqslant)$ be an ordered semiring and S a multiplicative subset of R. Then the relation*

$$\mathfrak{r} \;=\; \left\{ \left(\tfrac{x}{s}, \tfrac{y}{t}\right) \,\middle|\, wtx \leqslant wsy \text{ for some } w \in S \right\}$$

is an additive and multiplicative partial order on $S^{-1}R$. If $\frac{x}{s} = \frac{x'}{s'}$ and $\frac{y}{s} = \frac{y'}{t'}$, then $(\frac{x}{s}, \frac{y}{t}) \in \mathfrak{r}$ if and only if $(\frac{x'}{s'}, \frac{y'}{t'}) \in \mathfrak{r}$.

Proof. We begin with the proof of the latter claim. Consider $\frac{x}{s} = \frac{x'}{s'}$ and $\frac{y}{s} = \frac{y'}{t'}$. By the symmetry of the claimed property, it is enough to prove one implication. Let us assume that $(\frac{x}{s}, \frac{y}{t}) \in \mathfrak{r}$. The hypotheses mean that there are $u, v, w \in S$ such that $us'x = usx'$, $vt'y = vty'$ and $wtx \leqslant wsy$. Thus we obtain

$$uvwstt'x' \;=\; uvws'tt'x \;\leqslant\; uvwss't'y \;=\; uvwsts'y'.$$

Since $uvwst \in S$, this shows that $(\frac{x'}{s'}, \frac{y'}{t'}) \in \mathfrak{r}$ as claimed.

We turn to the proof of the former claim. The reflectivity of $\mathfrak{r}$ is obvious. To verify antisymmetry, assume that $(\frac{x}{s}, \frac{y}{t})$ and $(\frac{y}{t}, \frac{x}{s})$ are in $\mathfrak{r}$, i.e. $wtx \leqslant wsy$ and $w'sy \leqslant w'tx$ for some $w, w' \in S$. Then $ww'tx = ww'sy$ in R and $\frac{x}{s} = \frac{y}{t}$ in $S^{-1}R$, as desired.

To verify transitivity, assume that $(\frac{x}{s},\frac{y}{t})$ and $(\frac{y}{t},\frac{z}{u})$ are in $\mathfrak{r}$, i.e. $wtx \leqslant wsy$ and $w'uy \leqslant w'tz$ for some $w,w' \in S$. Then $ww'tux \leqslant ww'suy \leqslant ww'tsz$ and $(\frac{x}{s},\frac{z}{u})$ is in $\mathfrak{r}$ as desired.

To verify additivity and multiplicativity, assume that $(\frac{x}{s},\frac{y}{t})$ is in $\mathfrak{r}$, i.e. $wtx \leqslant wsy$ for some $w \in S$. We want to show that for every $\frac{z}{u} \in S^{-1}R$, the pairs $(\frac{x}{s}+\frac{z}{u},\frac{y}{t}+\frac{z}{u})$ and $(\frac{x}{s}\cdot\frac{z}{u},\frac{y}{t}\cdot\frac{z}{u})$ are in $\mathfrak{r}$. Using the additivity and multiplicativity of $\leqslant$, we see that $wu^2tx+wustz \leqslant wu^2sy+wustz$ and $wtuxz \leqslant wsuyz$, which shows that $(\frac{ux+sz}{su},\frac{uy+tz}{tu})$ and $(\frac{xz}{su},\frac{yz}{tu})$ are in $\mathfrak{r}$, as desired. □

Definition 5.9.2. Let B be an ordered blueprint, S a multiplicative subset of B and $\mathfrak{r}$ the partial order on $S^{-1}B^+$ defined in Lemma 5.9.1. The *localization of B at S* is the ordered blueprint

$$S^{-1}B \;=\; \big(S^{-1}B^\bullet, S^{-1}B^+, \mathfrak{r}\big).$$

Let $h \in B$ and $S = \{h^i\}_{i\in\mathbb{N}}$. The *localization of B at h* is $B[h^{-1}] = S^{-1}B$. Let $\mathfrak{p}$ be a prime m-ideal of B and $S = B-\mathfrak{p}$. The *localization of B at $\mathfrak{p}$* is $B_\mathfrak{p} = S^{-1}B_\mathfrak{p}$. If $S = B-\{0\}$ is a multiplicative subset of B, then we define the *fraction field of B* as $\operatorname{Frac} B = S^{-1}B$.

Note that the localization map $\iota_S^+ : B^+ \to S^{-1}B^+$ is a morphism $\iota_S : B \to S^{-1}B$ of ordered blueprints that sends $a \in B$ to $\frac{a}{1}$. We call ι_S the *localization map of $S^{-1}B$*. Note that

$$S^{-1}B \;=\; S^{-1}B^\bullet /\!\!/ \langle \textstyle\sum \frac{a_i}{1} \leqslant \sum \frac{b_j}{1} \,|\, \sum a_i \leqslant \sum b_j \text{ in } B\rangle,$$

i.e. we can consider $S^{-1}B$ as a quotient of $S^{-1}B^\bullet$.

Lemma 5.9.3. *Let B be an ordered blueprint, S a multiplicative subset of B and $\iota_S : B \to S^{-1}B$ the localization map. Let $f : B \to C$ be a morphism such that $f(S) \subset C^\times$. Then there exists a unique morphism $f_S : S^{-1}B \to C$ such that $f = f_S \circ \iota_S$.*

Proof. Assuming that f_S exists, then it necessarily satisfies $f_S(\frac{a}{s}) = f(s)^{-1}f(a)$, which shows that it is uniquely determined.

We are left with proving the existence of f_S. By Exercise 3.6.3, there is a unique morphism $f_S^\bullet : S^{-1}B^\bullet \to C^\bullet$ such that $f^\bullet = \iota_S^\bullet \circ f_S^\bullet$ where $\iota_S^\bullet : B^\bullet \to S^{-1}B^\bullet$ is the localization map. Composing $f_S^\bullet$ with the identity map $C^\bullet \to C$ yields a morphism $\tilde f_S : S^{-1}B^\bullet \to C$ of ordered blueprints. Since $f : B \to C$ is a morphism, every relation $\sum a_i \leqslant \sum b_j$ in B implies a relation $\sum f(a_i) \leqslant \sum f(b_j)$ in C. Therefore we can apply Proposition 5.3.2 to the quotient map $S^{-1}B^\bullet \to S^{-1}B$, which yields a morphism $f_S : S^{-1}B \to C$ with the desired properties. □

Example 5.9.4. Let B be an ordered blueprint and $B[T]$ be the free ordered blueprint over B in one variable T. We denote by $B[T^{\pm1}]$ the localization $S^{-1}B[T] = B[T][T^{-1}]$ of B at $S = \{T^i\}_{i\in\mathbb{N}}$.

Lemma 5.9.5. *Let B be an ordered blueprint, S a multiplicative subset and $\iota_S : B \to S^{-1}B$ the localization map. Let I be an (m/k-)ideal of B. Then*

$$S^{-1}I \;=\; \big\{\tfrac{a}{s} \in S^{-1}B \,\big|\, a \in I, s \in S\big\}$$

is the (m/k-)ideal of $S^{-1}B$ that is generated by $\iota_S(I)$.

Proof. Since the (m/k-)ideals of B correspond to the (m/k-)ideals of its algebraic core B^{core}, the claim follows immediately from Lemma 4.8.7. □

Proposition 5.9.6. *Let B be an ordered blueprint, S a multiplicative subset of B and $\iota_S : B \to S^{-1}B$ the localization map. Then the maps*

$$\begin{array}{ccc} \{\text{prime } m\text{-ideals } \mathfrak{p} \text{ of } B \text{ with } \mathfrak{p} \cap S = \emptyset\} & \longleftrightarrow & \{\text{prime } m\text{-ideals of } S^{-1}B\} \\ \mathfrak{p} & \xmapsto{\Phi} & S^{-1}\mathfrak{p} \\ \iota_S^{-1}(\mathfrak{q}) & \xleftarrow{\Psi} & \mathfrak{q} \end{array}$$

are mutually inverse bijections. A prime m-ideal $\mathfrak{p}$ of B with $\mathfrak{p} \cap S = \emptyset$ is a (k-)ideal if and only if $S^{-1}\mathfrak{p}$ is a (k-)ideal.

Proof. Since the prime (m/k-)ideals of B correspond to the prime (m/k-)ideals of its algebraic core B^{core}, the claim follows immediately from Proposition 4.8.8. □

Exercise 5.9.7. Let B be an ordered blueprint, S a multiplicative subset of B and $\mathfrak{p}$ a prime m-ideal of B that does not intersect S. Show that $\mathfrak{p}$ is a monomial ideal of B if and only if $S^{-1}\mathfrak{p}$ is a monomial ideal of $S^{-1}B$.

Residue fields

Let B be an ordered blueprint, $\mathfrak{p}$ a prime m-ideal of B and $S = B - \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the complement of the units of $S^{-1}B$ and therefore its unique maximal m-ideal.

Definition 5.9.8. Let B be an ordered blueprint and $\mathfrak{p}$ a prime m-ideal of B. The *residue field at* $\mathfrak{p}$ is the blueprint $k(\mathfrak{p}) = B_\mathfrak{p} /\!\!/ \mathfrak{c}(S^{-1}\mathfrak{p})$ where S is the complement of $\mathfrak{p}$ in B and $\mathfrak{c}(S^{-1}\mathfrak{p})$ is the congruence on $B_\mathfrak{p}^+$ that is generated by $S^{-1}\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime m-ideal of a blueprint B. Then the residue field at $\mathfrak{p}$ comes with a canonical morphism $B \to k(\mathfrak{p})$, which is the composition of the localization map $B \to B_\mathfrak{p}$ with the quotient map $B_\mathfrak{p} \to k(\mathfrak{p})$. Note that the residue field $k(\mathfrak{p})$ can be the trivial semiring in case that $\mathfrak{p}$ is not a k-ideal. More precisely, we have the following.

Corollary 5.9.9. *Let B be a blueprint, $\mathfrak{p}$ a prime m-ideal of B and $S = B - \mathfrak{p}$. Then the residue field $k(\mathfrak{p})$ is a blue field if $\mathfrak{p}$ is a k-ideal and trivial if not.*

Proof. This can be proven as the corresponding Corollary 4.8.10 for algebraic blueprints. We repeat the argument in brevity.

By Proposition 5.9.6, $\mathfrak{m} = S^{-1}\mathfrak{p}$ is the unique maximal ideal in $B_\mathfrak{p}$ and $B_\mathfrak{p}^\times = B_\mathfrak{p} - \mathfrak{m}$. Let $\pi : B_\mathfrak{p} \to k(\mathfrak{p})$ be the quotient map.

By Proposition 4.6.9, $\ker \pi = \pi^{-1}(0)$ is a k-ideal of B. Thus we have $\ker \pi = \mathfrak{m}$ if $\mathfrak{p}$ is a k-ideal and $\ker \pi = B_\mathfrak{p}$ if not. In the former case, $k(\mathfrak{p})$ is a blue field, in the latter case $k(\mathfrak{p}) = \{0\}$, as claimed. □

Exercise 5.9.10. Let B be a nontrivial ordered blueprint. Show that there exists a morphism $B \to k$ into a blue field k.

Compatibility with reflections and coreflections

Similar to in the case of algebraic blueprints, localizations commute with most constructions that we have encountered in this chapter. We leave the verification of these compatibilities as an exercise.

Exercise 5.9.11. Let B be an ordered blueprint and S a multiplicative subset of B that does not contain 0. Show that if B is an ordered semiring, an ordered blue field, integral, without zero divisors, idempotent, cancellative, monomial, totally positive, strictly conic, with unique weak inverses or reversible, then $S^{-1}B$ is so, too.

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Chapter 6

Valuations

With the formalism developed in the previous chapter, we are ready to give the general definition of a valuation, which recovers absolute values, norms, seminorms, Krull valuations and group characters as special cases. We also explain the relation to valuations in idempotent semirings and morphisms of hyperrings, and conclude this chapter with some novel examples. Most of the contents of this chapter stem from [Lor15].

6.1 Definition

Definition 6.1.1. Let B and S be two ordered blueprints. A *valuation of B in S* is a morphism $v^\bullet : B^\bullet \to S^\bullet$ between the underlying monoids that admits a morphism $\tilde{v} : B^{\mathrm{mon}} \to S^{\mathrm{pos}}$ such that the diagram

$$\begin{tikzcd} B^\bullet \arrow[r, "v^\bullet"] \arrow[d] & S^\bullet \arrow[d] \\ B^{\mathrm{mon}} \arrow[r, "\tilde{v}"] & S^{\mathrm{pos}} \end{tikzcd}$$

commutes. We write $v : B \to S$ for a valuation v of B in S.

Remark 6.1.2. We collect some first facts about valuations.

(1) Since the canonical morphism $S^\bullet \to S^{\mathrm{mon}}$ is a bijection, $\tilde{v}$ is uniquely determined by v. Conversely, v is uniquely determined by $\tilde{v}$ if $S \to S^{\mathrm{pos}}$ is a bijection. By Corollary 5.6.20 (5), this holds for strictly conic S, which is the case of most interest for us.

(2) Using Corollary 5.6.10 (3) to characterize the relations of S^{pos}, a valuation is the same as a map $v : B \to S$ such that

- $v(0) = 0$ and $v(1) = 1$;
- $v(ab) = v(a)v(b)$ for all $a, b \in B$;
- if $a \leqslant \sum b_j$ in B, then $v(a) + \sum c_k \leqslant \sum v(b_j)$ for some $c_k \in S$.

(3) Every morphism $v : B \to S$ is a valuation since the diagram

$$\begin{tikzcd} B^\bullet \arrow[rrr, "v^\bullet"] \arrow[d] & & & S^\bullet \arrow[d] \\ B^{\mathrm{mon}} \arrow[r] \arrow[rrr, bend left, "\tilde{v}"] & B \arrow[r, "v"] & S \arrow[r] & S^{\mathrm{pos}} \end{tikzcd}$$

commutes. Conversely, if B is monomial and S totally positive, then every valuation $v : B \to S$ is a morphism.

(4) From the description in (2), it is apparent that the composition of a valuation $v : B \to S$ with a morphism $f : S \to T$ of ordered blueprints yields a valuation $f \circ v : B \to T$. But in general, valuations are not closed under composition, cf. Exercise 6.1.3.

Exercise 6.1.3. Show that both the identity map $\mathbb{F}_1[T]/\!\!/\langle T \leqslant 1+1\rangle \to \mathbb{F}_1[T]/\!\!/\langle T+T \leqslant 1+1\rangle$ and the identity map $\mathbb{F}_1[T]/\!\!/\langle T+T \leqslant 1+1\rangle \to \mathbb{F}_1[T]$ are valuations, but the composition $\mathbb{F}_1[T]/\!\!/\langle T \leqslant 1+1\rangle \to \mathbb{F}_1[T]$ is not.

6.2 Seminorms

Definition 6.2.1. Let R be a ring. A *seminorm* on R is a multiplicative map $v : R \to \mathbb{R}_{\geqslant 0}$ with $v(0) = 0$ and $v(1) = 1$ that satisfies the *triangle inequality* $v(a+b) \leqslant v(a)+v(b)$ for all $a,b \in R$. A *nonarchimedean seminorm* is a monoid morphism that satisfies the *strong triangle inequality* $v(a+b) \leqslant \max\{v(a),v(b)\}$.

Lemma 6.2.2. *A map $v : R \to \mathbb{R}_{\geqslant 0}$ is a seminorm if and only if it is a valuation. It is a nonarchimedean seminorm if and only if the composition $R \to \mathbb{R}_{\geqslant 0} \to \mathbb{T}$ of v with the identity map $\mathbb{R}_{\geqslant 0} \to \mathbb{T}$ is a valuation.*

Proof. The properties that $v(0) = 0$, $v(1) = 1$ and $v(ab) = v(a)v(b)$ for all a and b are common to semirings and valuations. We have to verify that the conditions on sums agree for both types of maps.

Assume that v is a seminorm and consider $a \leqslant \sum b_j$ in R. Since R is trivially ordered, this means that $a = \sum b_j$. Since R is a ring, it contains all partial sums of $\sum b_j$ and we can use the triangle inequality recursively to obtain $v(a) \leqslant \sum v(b_j)$ with respect to the natural total order of $\mathbb{R}_{\geqslant 0}$. This means that the difference $c = \sum v(b_j) - v(a)$ is positive and in $\mathbb{R}_{\geqslant 0}$ and $v(a)+c = \sum v(b_j)$ in $\mathbb{R}_{\geqslant 0}$. Thus v is a valuation.

Assume that v is a valuation and consider $c = a+b$ in R. Then $c \leqslant a+b$ in R and $v(c)+\sum d_l \leqslant v(a)+v(b)$ for some $d_l \in \mathbb{R}_{\geqslant 0}$. This shows that $v(c) \leqslant v(a)+v(b)$ with respect to the natural total order of $\mathbb{R}_{\geqslant 0}$ and that v is a seminorm. Thus the former claim of the lemma.

Since the addition of the tropical semiring $\mathbb{T}$ is $a+b = \max\{a,b\}$, the latter claim of the lemma follows by the same argument as the former claim. □

Remark 6.2.3. The nonarchimedean seminorms can be characterized as the following seminorms. By the universal property of a monomial blueprint, a morphism $R^{\mathrm{mon}} \to \mathbb{R}^{\mathrm{pos}}_{\geqslant 0}$ factorizes uniquely into $R^{\mathrm{mon}} \to (\mathbb{R}^{\mathrm{pos}}_{\geqslant 0})^{\mathrm{mon}} \to \mathbb{R}^{\mathrm{pos}}_{\geqslant 0}$. Using the morphism $(\mathbb{T}^{\mathrm{pos}})^{\mathrm{mon}} \to (\mathbb{R}^{\mathrm{pos}}_{\geqslant 0})^{\mathrm{mon}}$ from Exercise 5.6.24, we see that the seminorm $v : R \to \mathbb{R}_{\geqslant 0}$ is nonarchimedean if and only if $R^{\mathrm{mon}} \to \mathbb{R}^{\mathrm{pos}}_{\geqslant 0}$ factors into

$$R^{\mathrm{mon}} \longrightarrow (\mathbb{T}^{\mathrm{pos}})^{\mathrm{mon}} \longrightarrow (\mathbb{R}^{\mathrm{pos}}_{\geqslant 0})^{\mathrm{mon}} \longrightarrow \mathbb{R}^{\mathrm{pos}}_{\geqslant 0}.$$

6.3 Krull valuations

A *partially ordered monoid with zero* is an ordered blueprint Γ_0 whose algebraic core is a monoid, i.e. $\Gamma_0^+ = (\Gamma_0^\bullet)^+$, and whose partial order $\leqslant$ is generated by relations of the from $a \leqslant b$ with $a,b \in \Gamma_0$. Note that $\Gamma_0^\bullet$ together with the restriction of $\leqslant$ to $\Gamma_0^\bullet$ is indeed a partially ordered monoid with zero in the proper sense, which justifies our abuse of language. Often we want that 0 is the smallest element of Γ_0 or, in other words, that Γ_0 is totally positive.

Proposition 6.3.1. *Let Γ_0 be a totally positive partially ordered monoid with zero. Then the following holds true.*

(1) The tensor product $\Gamma_{\mathbb{B}} = \Gamma_0 \otimes_{\mathbb{F}_1} \mathbb{B}$ is a totally positive blueprint with idempotent algebraic core $\Gamma_{\mathbb{B}}^{\mathrm{core}} = (\Gamma_{\mathbb{B}})^{\mathrm{core}}$.

(2) The canonical morphism $\iota_{\mathbb{B}} : \Gamma_0 \to \Gamma_{\mathbb{B}}$ is bijective and $a \leqslant b$ in Γ_0 if and only if $\iota_{\mathbb{B}}(a) \leqslant \iota_{\mathbb{B}}(b)$ in $\Gamma_{\mathbb{B}}$.

(3) The canonical morphism $(\Gamma_{\mathbb{B}}^{\mathrm{core}})^{\mathrm{pos}} \to \Gamma_{\mathbb{B}}$ is an isomorphism.

(4) If Γ_0 is totally ordered, then $\Gamma_{\mathbb{B}}^{\mathrm{core}}$ is a bipotent semiring with $a+b = \max\{a,b\}$.

Proof. Since $0 \leqslant 1$ in Γ_0, this also holds in $\Gamma_{\mathbb{B}}$, which shows that $\Gamma_{\mathbb{B}}$ is totally positive. Since $1+1 = 1$ in $\mathbb{B}$, this also holds in $\Gamma_{\mathbb{B}}$, which shows that its algebraic core is idempotent. Thus (1).

Since $\Gamma_{\mathbb{B}} = \Gamma_0 /\!\!/ \langle 1+1 \equiv 1 \rangle$, it is clear that $\iota_{\mathbb{B}}$ is surjective and that $a \leqslant b$ in Γ_0 implies $\iota_{\mathbb{B}}(a) \leqslant \iota_{\mathbb{B}}(b)$ in $\Gamma_{\mathbb{B}}$. The injectivity of $\iota_{\mathbb{B}}$ follows from the fact that $\iota_{\mathbb{B}}(a) \leqslant \iota_{\mathbb{B}}(b)$ implies $a \leqslant b$, as proven next.

Consider $\iota_{\mathbb{B}}(a) \leqslant \iota_{\mathbb{B}}(b)$ in $\Gamma_{\mathbb{B}}$. Then this relation is derived from the generators of the partial order of $\Gamma_{\mathbb{B}}^{+}$ by using transitivity, additivity and multiplicativity. The generators are of the form $1+1 = 1$ and $\iota_{\mathbb{B}}(c) \leqslant \iota_{\mathbb{B}}(d)$ where $c \leqslant d$ in Γ_0. Each of the generators is a relation of the form $\sum \iota_{\mathbb{B}}(c_k) \leqslant \sum \iota_{\mathbb{B}}(d_l)$ with the property that there is an l for every k such that $c_k \leqslant d_l$. Since this property stable under transitive, additive and multiplicative closures, we conclude that $a \leqslant b$ in Γ_0. Thus (2).

Let $a \leqslant b$ be a relation in Γ_0. Then we have $b = 0+b \leqslant a+b \leqslant b+b = b$ in $\Gamma_{\mathbb{B}}$ and $a+b = b$ in its algebraic core $\Gamma_{\mathbb{B}}^{\mathrm{core}}$. Using Corollary 5.6.10 (3), we conclude that $a \leqslant b$ in $(\Gamma_{\mathbb{B}}^{\mathrm{core}})^{\mathrm{pos}}$. Note that $1+1 = 1$ holds in $\Gamma_{\mathbb{B}}^{\mathrm{core}}$ and thus also in $(\Gamma_{\mathbb{B}}^{\mathrm{core}})^{\mathrm{pos}}$. Since the partial order of $\Gamma_{\mathbb{B}}$ is generated by relations of the form $a \leqslant b$ and $1+1 = 1$, the canonical morphism $(\Gamma_{\mathbb{B}}^{\mathrm{core}})^{\mathrm{pos}} \to \Gamma_{\mathbb{B}}$ is an isomorphism. Thus (3)

If Γ is totally ordered, then for all a and b, the sum $a+b = \max\{a,b\}$ is defined by the above argument. Therefore $\Gamma_{\mathbb{B}}^{\mathrm{core}}$ is a bipotent semiring. Thus (4). □

Let Γ be a multiplicatively written partially ordered commutative semigroup with unit 1. We denote by Γ_0 the ordered blueprint $(\Gamma \cup \{0\}, \mathcal{R})$ where $\mathcal{R}$ is generated by the partial order of Γ and the relation $0 \leqslant 1$. Note that Γ_0 is a totally positive ordered monoid with zero.

Definition 6.3.2. Let k be a field and Γ a totally ordered group. A *Krull valuation of k with value group Γ* is a map $v : k \to \Gamma_0$ with $v(0) = 0$, $v(1) = 1$, $v(ab) = v(a)v(b)$ and $v(a+b) \leqslant \max\{v(a), v(b)\}$ for all $a, b \in k$.

Corollary 6.3.3. *A map $v : k \to \Gamma_0$ is a Krull valuation if and only if the composition*

$$\tilde{v} : \; k^{\mathrm{mon}} \longrightarrow k \xrightarrow{\;v\;} \Gamma_0 \xrightarrow{\;\iota_{\mathbb{B}}\;} \Gamma_{\mathbb{B}}$$

is a morphism of ordered blueprints. In other words, the map $a \mapsto v(a)$ is a valuation $k \to \Gamma_{\mathbb{B}}^{\mathrm{core}}$.

Proof. Since $k^{\mathrm{mon}} \to k$ and $\Gamma_0 \to \Gamma_{\mathbb{B}}$ are bijections, cf. Proposition 6.3.1, v is a monoid morphism if and only if $\tilde{v}$ is so. Thus we only have to verify the respective properties for sums in the following.

Assume that v is a Krull valuation and consider a relation $a \leqslant \sum b_j$ in k^{mon}. Then $a = \sum b_j$ in k since k is algebraic. Since v is a valuation and Γ_0 is totally ordered, we can use inductively that $v(c+d) \leqslant \max\{v(c), v(d)\}$ to show that $a \leqslant \max\{v(b_j)\}$ in Γ_0. Thus $\tilde{v}(a) \leqslant \sum \tilde{v}(b_j)$ in $\Gamma_{\mathbb{B}}$. This shows that $\tilde{v}$ is a morphism.

Conversely assume that $\tilde{v}$ is a morphism and consider $c = a+b$ in k. Then $c \leqslant a+b$ in k^{mon} and $\tilde{v}(c) \leqslant \tilde{v}(a)+\tilde{v}(b)$ in $\Gamma_{\mathbb{B}}$. Then $\tilde{v}(c) = \iota_{\mathbb{B}}(v(c))$ and $\tilde{v}(a)+\tilde{v}(b) = \iota_{\mathbb{B}}\big(\max\{v(a), v(b)\}\big)$. By Proposition 6.3.1 (2), we have that $v(c) \leqslant \max\{v(a), v(b)\}$. This proves that v is a Krull valuation and completes the proof. □

Remark 6.3.4. In the original definition of a Krull valuation, the group Γ is written additively, the additional element 0 is denoted by ∞ and the triangle inequality is expressed with respect to the reverse order on Γ, i.e. $v(a+b) \geqslant \min\{v(a), v(b)\}$. Since the group operation of Γ is the multiplication of the associated semiring $\Gamma_{\mathbb{B}}$ and the additional element is the zero of $\Gamma_{\mathbb{B}}$, we allow ourselves to break with precedent and adopt the notation to fit readily into our formalism.

According to Proposition 6.3.1, the concept of a Krull valuation can be generalized by considering seminorms of R in idempotent semirings S, which correspond to morphisms $R^{\text{mon}} \to S^{\text{pos}}$ of ordered blueprints. We will see in a later chapter that the class of idempotent semirings plays a particular role for tropicalizations. This viewpoint can also be found in Macpherson's paper [Mac13].

6.4 Valuations into idempotent semirings

By Corollary 6.3.3, a Krull valuation corresponds to a valuation in an idempotent semiring. It is a well-known theme to consider the more general class of valuations of rings in idempotent semirings, which turn out to be particular instances of the more general notion developed in this chapter; for instance cf. [GG16] and [Mac13]. In this section, we review one particular result for valuations in idempotent semirings, which is the existence of a universal such valuation.

Let R be a semiring. Let Γ_R be the idempotent semiring that consists of all finitely generated ideals I of R together with the product and sum of ideals. Let $\langle a\rangle$ be the principal ideal generated by $a \in R$. Then $\langle 0\rangle$ is the zero and that $\langle 1\rangle$ is the one of Γ_R, and we have $\langle a\rangle \cdot \langle b\rangle = \langle ab\rangle$ for all $a, b \in R$. Note that $\langle a\rangle = \langle b\rangle$ if and only if $b = ua$ for a unit $u \in R^\times$.

We call a valuation $v : R \to S$ into an idempotent semiring S *integral* if $v(a)+1 = 1$ for every $a \in R$. Note that this is the same as $v(a) \leqslant 1$ in S^{pos}, by Exercise 5.1.5.

Lemma 6.4.1. *Let R be a semiring. The map $v : R \to \Gamma_R$ that sends an element a of R to the principal ideal $\langle a\rangle$ is an integral valuation. For any integral valuation $w : R \to S$ into an idempotent blueprint S, there exists a unique semiring morphism $\tilde{w} : \Gamma_R \to S$ such that $w = \tilde{w} \circ v$ as maps.*

Proof. We verify all properties for v listed in Remark 6.1.2 (2). It follows from our previous observations that $v(0) = 0$, $v(1) = 1$ and $v(ab) = v(a)v(b)$ for all $a, b \in R$. Given a relation $a \leqslant \sum b_j$ in R, then $a = \sum b_j$ since R is algebraic and thus a is contained in the ideal $\langle b_j\rangle$ generated by the b_j. Therefore we have an equality $v(a)+\sum v(b_j) = \langle a, b_j\rangle = \langle b_j\rangle = \sum v(b_j)$ in Γ_B^+. This shows that v is a valuation. It is integral since $v(a)+1 = \langle a, 1\rangle = \langle 1\rangle = 1$ for every $a \in R$.

Given a valuation $w : R \to S$ into an idempotent semiring, the condition $w = \tilde{w} \circ v$ prescribes that $\tilde{w}(\langle a_i\rangle) = \sum w(a_i)$, which implies the uniqueness of $\tilde{w}$. Assuming that $\tilde{w}$ is well-defined, it is clear from this definition that $\tilde{w}$ is a semiring morphism.

We verify that the definition of $\tilde{w}$ does not depend on the choice of generators. If I is an ideal of R that is generated by two different finite subsets, then it is also generated by their union. Thus we may reduce the proof to the situation that one of the subsets is contained in the other subset. By joining the elements of the larger subset one by one, we can reduce this subsequently to the case that the subsets differ by cardinality one. Thus assume that $\langle b_j \rangle = \langle a, b_j \rangle$ for some $a \in \langle b_j \rangle$. We need to show that $w(a) + \sum w(b_j) = \sum w(b_j)$ in S.

By Corollary 2.5.4, there are elements $x_j \in R$ such that $a = \sum x_j b_j$. Since w is a valuation, we have $w(a) + \sum c_k = \sum w(x_j) w(b_j)$ for some $c_k \in S$; cf. Remark 6.1.2 (2). Since S is idempotent, we have in fact that

$$w(a) + \sum w(x_j) w(b_j) \ = \ w(a) + w(a) + \sum c_k \ = \ w(a) + \sum c_k \ = \ \sum w(x_j) w(b_j).$$

Note that since w is integral, we have $w(x_j) + 1 = 1$. Thus adding $\sum 1 \cdot w(b_j)$ to both sides of the displayed equation yields $w(a) + \sum w(b_j) = \sum w(b_j)$, as desired. This completes the proof of the lemma. $\square$

Exercise 6.4.2. Extend Lemma 6.4.1 to ordered blueprints as follows. Let B be an ordered blueprint. We call a valuation $v : B \to S$ into an idempotent ordered blueprint *integral* if $v(a) + 1 = 1$ for every $a \in B$. Recall from section 5.8 that a monomial ideal of B is an m-ideal I such that $a \leqslant \sum b_j$ with $b_j \in I$ implies $a \in I$.

Show that for every subset S of B, there exists a unique smallest monomial ideal I containing S. We call I the *monomial ideal generated by* S and write $I = \langle S \rangle_{\mathrm{mon}}$. We say that a monomial ideal I is *principal* if it is generated by one element, i.e. $I = \langle a \rangle$ for some $a \in B$. We say that I is *finitely generated* if it is generated by a finite subset, i.e. $I = \langle a_1, \dots, a_n \rangle$ for some $a_1, \dots, a_n \in B$.

The product and sum of monomial ideals are defined as the monomial ideals generated by the pairwise sums and products, respectively, of elements of the ideals. Show that $\langle a \rangle \cdot \langle b \rangle = \langle ab \rangle$. Show that $\langle a \rangle = \langle b \rangle$ if and only if $b = ua$ for some unit $b \in B^\times$. Conclude that the set A of all principal monomial ideals of B together with the product of monomial ideals is a monoid with zero and that the map $v^\bullet : B^\bullet \to A$ sending a to $\langle a \rangle$ is a surjective morphism of monoids with zero, which identifies $B/B^\times$ with A.

We define the ordered blueprint

$$\Gamma'_B = A /\!\!/ \langle v^\bullet(a) \leqslant \sum v^\bullet(b_j) \,|\, a \in \langle b_j \rangle \rangle.$$

Show that $v^\bullet$ defines an integral valuation $v : B \to \Gamma'_B$. Show further that for every integral valuation $w : B \to S$ into an idempotent ordered blueprint S, there is a unique morphism of ordered blueprints $\tilde{w} : \Gamma'_B \to S$ such that $w = \tilde{w} \circ v$ as maps.

6.5 Characters

Definition 6.5.1. Let G be an abelian group and k a field. A *character of G in k* is a group homomorphism $\chi : G \to k^\times$. A *unitary character* is a character $\chi : G \to \mathbb{C}^\times$ whose image is contained in the unit circle $\mathbb{S}^1$.

As in the previous section, we write the group G multiplicatively and define by $G_0 = G \cup \{0\}$ its extension to a monoid with zero where $0 \cdot a = 0$ for all $a \in G_0$. The monoid $\mathbb{S}^1 = \mathbb{S}^1 \cup \{0\}$ inherits a nontrivial structure of a blueprint from the embedding to $\mathbb{C}$—we define

$$\mathbb{S}^1_{0,\mathbb{C}} \ = \ \mathbb{S}^1_0 /\!\!/ \langle \sum a_i = \sum b_j \,|\, \sum a_i = \sum b_j \text{ in } \mathbb{C} \rangle.$$

Lemma 6.5.2. *The maps*

$$\begin{array}{ccc} \{\,\textit{valuations } v: G_0 \to k\,\} & \longrightarrow & \{\,\textit{characters } \chi: G \to k^\times\,\} \\ v: G_0 \to k & \longmapsto & v^\bullet|_G : G \to k^\times \end{array}$$

and

$$\begin{array}{ccc} \{\,\textit{valuations } v: G_0 \to \mathbb{S}^1_{0,\mathbb{C}}\,\} & \longrightarrow & \{\,\textit{unitary characters } \chi: G \to \mathbb{C}^\times\,\} \\ v: G_0 \to \mathbb{S}^1_{0,\mathbb{C}} & \longmapsto & v^\bullet|_G : G \to \mathbb{C}^\times \end{array}$$

are bijections.

Proof. Since k contains an additive inverse -1 of 1, we have $k^{\mathrm{pos}} = \{0\}$, cf. Corollary 5.6.10 (2). Thus a valuation $v: G_0 \to k$ is nothing else than a morphism $G_0 \to k^\bullet$ of monoids with zero.

Since every nonzero element of G_0 is invertible, we have $v(G) \subset k^\times$, which shows that $v^\bullet|_G : G \to k^\times$ is well-defined as a map. Conversely, we can extend every character $G \to k^\times$ uniquely to a map $v: G_0 \to k$ with $v(0) = 0$. It is evident that $v: G_0 \to k$ is multiplicative if and only if $v^\bullet|_G : G \to k^\times$ is multiplicative. This establishes the first bijection.

Since a unitary character of G is the same as a group homomorphism $G \to \mathbb{S}^1$, the same reasoning establishes the second bijection. □

6.6 Nonarchimedean seminorms as hyperring morphisms

An alternative viewpoint to the approach in the previous sections is the interpretation of nonarchimedean seminorms as hyperfield morphisms. This interpretation is already implicit in Viro's paper [Vir11] about tropical geometry in the language of hyperfields, and was made explicit in [BB16].

The precise statement is as follows. Let R be a ring and R^{hyp} the corresponding hyperring whose hyperaddition is given by $a \boxplus b = \{a+b\}$. Let $\mathbb{T}^{\mathrm{hyp}}$ be the tropical hyperfield whose hyperaddition is given by $a \boxplus b = \{\max\{a,b\}\}$ if $a \neq b$ and $a \boxplus a = [0,a]$. In the following, we use the identifications $R = R^{\mathrm{hyp}}$ and $\mathbb{R}_{\geqslant 0} = \mathbb{T}^{\mathrm{hyp}}$ as sets.

Proposition 6.6.1. *A map $v: R \to \mathbb{R}_{\geqslant 0}$ is a nonarchimedean seminorm if and only if $v: R^{\mathrm{hyp}} \to \mathbb{T}^{\mathrm{hyp}}$ is a morphism of hyperrings.*

We will reprove this fact in the language of ordered blueprints. The ordered blueprints associated with R^{hyp} and $\mathbb{T}^{\mathrm{hyp}}$ are

$$(R^{\mathrm{hyp}})^{\mathrm{ob}} \;=\; R^\bullet /\!\!/ \langle c \leqslant a+b \,|\, c \in a \boxplus b \rangle \qquad \text{and} \qquad (\mathbb{T}^{\mathrm{hyp}})^{\mathrm{ob}} = \mathbb{T}^\bullet /\!\!/ \langle c \leqslant a+b \,|\, c \in a \boxplus b \rangle.$$

Since $(-)^{\mathrm{ob}} : \mathrm{HypRings} \to \mathrm{OBlpr}$ is a fully faithful embedding, cf. Exercise 5.7.5, we may identify T^{hyp} with $(\mathbb{T}^{\mathrm{hyp}})^{\mathrm{ob}}$ to avoid an overcharged notation. Note that $(R^{\mathrm{hyp}})^{\mathrm{ob}} = R^{\mathrm{mon}}$.

Lemma 6.6.2. *We have $\mathbb{T}^{\mathrm{hyp}} /\!\!/ \langle 1+1 \equiv 1 \rangle = \mathbb{T}^{\mathrm{pos}}$. In particular, the identity map is a morphism $\mathbb{T}^{\mathrm{hyp}} \to \mathbb{T}^{\mathrm{pos}}$.*

Proof. Consider a relation $c \leqslant a+b$ in $\mathbb{T}^{\mathrm{hyp}}$. If $c = \max\{a,b\}$ with respect to the natural total order on $\mathbb{T}$, then $c = a+b$ in $\mathbb{T}$ and thus in $\mathbb{T}^{\mathrm{pos}}$. If $c \leqslant a = b$, then this is the case in $\mathbb{T}^{\mathrm{pos}}$. Thus the identity map is a morphism $\mathbb{T}^{\mathrm{hyp}} \to \mathbb{T}^{\mathrm{pos}}$.

Clearly, $1+1 = 1$ holds in $\mathbb{T}^{\mathrm{hyp}}$. Thus we are left with verifying that every relation of $\mathbb{T}^{\mathrm{pos}}$ holds already in $\mathbb{T}^{\mathrm{hyp}} /\!\!/ \langle 1+1 = 1 \rangle$. To begin with, we have $0 \leqslant 1+1 = 1$ in $\mathbb{T}^{\mathrm{hyp}} /\!\!/ \langle 1+1 = 1 \rangle$. Given an equality $a+b = b$ in $\mathbb{T}^{\mathrm{pos}}$, then $a+b \leqslant b+b = b$ in $\mathbb{T}^{\mathrm{hyp}} /\!\!/ \langle 1+1 = 1 \rangle$. Since the partial order of $\mathbb{T}^{\mathrm{pos}}$ is generated by these two types of relations, this completes the proof of the lemma. □

Exercise 6.6.3. Let R be a blueprint with inverses and S be a bipotent semiring, i.e. $x+y \in \{x,y\}$ for all $x,y \in S$. Let $v : R \to S$ be a valuation and $a \in R$ satisfy $a^n = 1$ for some $n \geqslant 1$. Show that $v(a) = 1$. In particular, show that $v(-1) = 1$.

Let $v : R \to S$ be a valuation and $a = \sum b_j$ a relation in R. Assume that there is a j such that $v(b_j) > v(b_i)$ for all $i \neq j$. Show that $v(a) = v(b_j)$. In particular, we have $v(a+b) = \max\{v(a), v(b)\}$ whenever $a+b$ is in R and $v(a) \neq v(b)$.

Since a valuation $v : R \to \mathbb{T}$ is nothing else than a morphism $\tilde{v} : R^{\mathrm{mon}} \to \mathbb{T}^{\mathrm{pos}}$ of ordered blueprints, cf. Remark 6.1.2 (1), the following lemma recovers Proposition 6.6.1.

Lemma 6.6.4. *Let R be a ring. Then every morphism $\tilde{v} : R^{\mathrm{mon}} \to \mathbb{T}^{\mathrm{pos}}$ factors into a uniquely determined morphism $\tilde{v}^{\mathrm{hyp}} : R^{\mathrm{mon}} \to T^{\mathrm{hyp}}$ composed with $\mathbb{T}^{\mathrm{hyp}} \to \mathbb{T}^{\mathrm{pos}}$.*

Proof. The partial order of R^{mon} is generated by relations of the form $c \leqslant a+b$. Since $\tilde{v}(c) \leqslant \tilde{v}(a) = \tilde{v}(b)$ holds in $\mathbb{T}^{\mathrm{hyp}}$, we are left with the case $\tilde{v}(a) \neq \tilde{v}(b)$. By Exercise 6.6.3, we have $\tilde{v}(a+b) = \max\{\tilde{v}(a), \tilde{v}(b)\}$ in this case, which also holds in $\mathbb{T}^{\mathrm{hyp}}$. This proves that $\tilde{v}$ factors through $\mathbb{T}^{\mathrm{hyp}}$. The uniqueness of this factorization is clear since $\mathbb{T}^{\mathrm{hyp}} \to \mathbb{T}^{\mathrm{pos}}$ is a bijection. $\square$

Exercise 6.6.5. Prove the following generalization of Lemma 6.6.4. Let R be a blueprint with inverses, S a bipotent semiring and S^{hyp} the associated hyperring from Exercise 5.7.8, considered as an ordered blueprint. Show that the identity map defines a morphism $S^{\mathrm{hyp}} \to S^{\mathrm{pos}}$ and that every morphism $\tilde{v} : R^{\mathrm{mon}} \to \mathbb{T}^{\mathrm{pos}}$ factors into a uniquely determined $\tilde{v}^{\mathrm{hyp}} : R^{\mathrm{mon}} \to T^{\mathrm{hyp}}$ composed with $S^{\mathrm{hyp}} \to S^{\mathrm{pos}}$.

6.7 More examples of valuations

We present some more examples of valuations in sense of this chapter.

Example 6.7.1. As we have seen in the previous sections, the identity maps $\mathbb{T} \to \mathbb{R}_{\geqslant 0}$ and $\mathbb{T}^{\mathrm{hyp}} \to \mathbb{T}$ are valuations; cf. Remark 6.2.3 and Lemma 6.6.2, respectively.

Example 6.7.2. The archimedean absolute value $v : \mathbb{Q} \to \mathbb{R}_{\geqslant 0}$, which sends x to $v(x) = \mathrm{sign}(x) \cdot x$, restricts to a valuation $\mathbb{Z} \to \mathbb{N}$ and to a valuation $\mathbb{F}_{1^2} \to \mathbb{F}_1$. The composition with the unique morphism $\mathbb{F}_1 \to \mathbb{B}$ yields a valuation $\mathbb{F}_{1^2} \to \mathbb{B}$.

Exercise 6.7.3. Show that every nontrivial ordered blueprint B admits a valuation $v : B \to \mathbb{B}$.

Exercise 6.7.4. Let $\mathbb{N}_{\mathrm{gcd}}$ be the idempotent semiring whose underlying monoid is $\mathbb{N}^\bullet$ and whose addition is given by $a+b = \gcd(a,b)$. Show that $x \leqslant y$ in $\mathbb{N}_{\mathrm{gcd}}^{\mathrm{pos}}$ if y divides x. Show that the map $x \mapsto \mathrm{sign}(x) \cdot x$ defines a valuation $v : \mathbb{Z} \to \mathbb{N}_{\mathrm{gcd}}$.

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