GEOMETRY OVER THE FIELD WITH ONE ELEMENT

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1. MOTIVATION

Two main sources have led to the development of several notions of F_1-geometry in the recent five years. We will concentrate on one of these, which originated as remark in a paper by Jacques Tits ([10]). For a wide class of schemes X (including affine space $\mathbb{A}^n$, projective space $\mathbb{P}^n$, the Grassmannian $\mathrm{Gr}(k,n)$, split reductive groups G), the function

$$N(q) \;=\; \#X(\mathbb{F}_q)$$

is described by a polynomial in q with integer coefficients, whenever q is a prime power. Taking the value $N(1)$ sometimes gives interesting outcomes, but has a 0 of order r in other cases. A more interesting number is the lowest non-vanishing coefficient of the development of $N(q)$ around $q-1$, i.e. the number

$$\lim_{q\to 1} \frac{N(q)}{(q-1)^r},$$

which Tits took to be the number $\#X(\mathbb{F}_1)$ of "$\mathbb{F}_1$-points" of X. The task at hand is to extend the definition of the above mentioned schemes X to schemes that are "defined over $\mathbb{F}_1$" such that their set of $\mathbb{F}_1$-points is a set of cardinality $\#X(\mathbb{F}_1)$. We describe some cases, and suggest an interpretation of the set of $\mathbb{F}_1$-points:

- $\#\mathbb{P}^{n-1}(\mathbb{F}_1) = n = \#M_n$ with $M_n := \{1,\dots,n\}$.
- $\#\,\mathrm{Gr}(k,n)(\mathbb{F}_1) = \binom{n}{k} = \#M_{k,n}$ with $M_{k,n} = \{\text{subsets of } M_n \text{ with } k \text{ elements}\}$.
- If G is a split reductive group of rank r, $T \simeq \mathbb{G}_m^r \subset G$ is a maximal torus, N its nomalizer and $W = N(\mathbb{Z})/T(\mathbb{Z})$, then the Bruhat decomposition $G(\mathbb{F}_q) = \coprod_{w\in W} BwB(\mathbb{F}_q)$ (where B is a Borel subgroup containing T) implies that $N(q) = \sum_{w\in W}(q-1)^r q_w^d$ for certain $d_w \geq 0$. This means that $\#G(\mathbb{F}_1) = \#W$.

In particular, it is natural to ask whether the group law $m : G\times G \to G$ of a split reductive group may be defined as a "morphism over $\mathbb{F}_1$". If so, one can define "group actions over $\mathbb{F}_1$". The limit as $q \to 1$ of the action

$$\mathrm{GL}(n,\mathbb{F}_q) \times \mathrm{Gr}(k,n)(\mathbb{F}_q) \longrightarrow \mathrm{Gr}(k,n)(\mathbb{F}_q)$$

induced by the action on $\mathbb{P}^{n-1}(\mathbb{F}_q)$ should be the action

$$S_n \times M_{k,n} \longrightarrow M_{k,n}$$

induced by the action on $M_n = \{1,\dots,n\}$.

The other, more lofty motivation for $\mathbb{F}_1$-geometry stems from the search for a proof of the Riemann hypothesis. In the early 90s, Deninger gave criteria for a category of motives that would provide a geometric framework for translating Weil's proof of the Riemann hypothesis for global fields of positive characteristic to number fields. In particular, the Riemann zeta function $\zeta(s)$ should have a cohomological interpretation, where an H^0, an H^1 and an H^2-term are involved. Manin proposed in [7] to interpret the H^0-term as the zeta function of the "absolute point" $\mathrm{Spec}\,\mathbb{F}_1$ and the H^2-term as the zeta function of the "absolute Tate motive" or the "affine line over $\mathbb{F}_1$".

2. Overview over recent approaches

We give a rough description of the several approaches towards $\mathbb{F}_1$-geometry, some of them looking for weaker structures than rings, e.g. monoids, others looking for a category of schemes with certain additional structures. In the following, a *monoid* always means a abelian mutliplicative semi-group with 1. A *variety* is a scheme X that defines, via base extension, a variety X_k over any field k.

2.1. **Soulé, 2004 (**[9]**).** This is the first paper that suggests a candidate of a category of varieties over $\mathbb{F}_1$. Soulé consideres schemes together with a complex algebra, a functor on finite rings that are flat over $\mathbb{Z}$ and certain natural transformations and a universal property that connects the scheme, the functor and the algebra. Soulé could prove that smooth toric varieties provide natural examples of $\mathbb{F}_1$-varieties. In [6] the list of examples was broadened to contain models of all toric varieties over $\mathbb{F}_1$, as well as split reductive groups. However, it seems unlikely that Grassmannians that are not projective spaces can be defined in this framework.

2.2. **Connes-Consani, 2008 (**[1]**).** The approach of Soulé was modified by Connes and Consani in the following way. They consider the category of scschemes together with a functor on finite abelian groups, a complex variety, certain natural transformations and a universal property analogous to Soulé's idea. This category behaves only slightly different in some subtle details, but the class of established examples is the same (cf. [6]).

2.3. **Deitmar, 2005 (**[3]**).** A completely different approach was taken by Deitmar who uses the theory of prime ideals of monoids to define spectra of monoids. A $\mathbb{F}_1$-scheme is a topological space together with a sheaf of monoids that is locally isomorphic the spectrum of a ring. This theory has the advantage of having a very geometric flavour and one can mimic algebraic geometry to a large extent. However, Deitmar has shown himself in a subsequent paper that the $\mathbb{F}_1$-schemes whose base extension to $\mathbb{Z}$ are varieties are nothing more than toric varieties.

2.4. **Toën-Vaquié, 2008 (**[11]**).** Deitmar's approach is complemented by the work of Toën and Vaquié, which proposes locally representable functors on monoids as $\mathbb{F}_1$-schemes. Marty shows in [8] that the Noetherian $\mathbb{F}_1$-schemes in Deitmar's sense correspond to the Noetherian objects in Toën-Vaquié's sense. We raise the question: is the Noetherian condition necessary?

2.5. **Borger, in progress.** The category investigated by Borger are schemes X together with a family of morphism $\{\psi_p : X \to X\}_{p \text{ prime}}$, where the ψ_p's are lifts of the Frobenius morphisms $\mathrm{Frob}_p : X \otimes \mathbb{F}_p \to X \otimes \mathbb{F}_p$ and all ψ_p's commute with each other.

There are further approaches by Durov ([4], 2007) and Haran ([5], 2007), which we do not describe here. In the following section we will examine more closely a new framework for $\mathbb{F}_1$-geometry by Connes and Consani in spring 2009.

3. $\mathbb{F}_1$-schemes à la Connes-Consani and torified varieties

The new notion of an $\mathbb{F}_1$-scheme due to Connes and Consani ([2]) combines the earlier approaches of Soulé and of themselves with Deitmar's theory of spectra of monoids and Toën-Vaquié's functorial viewpoint. First of all, Connes and Consani consider monoids with 0 and remark that the spaces that are locally isomorphic to spectra of monoids with 0, called *M_0-schemes*, are the same as locally representable functors of monoids with 0. (Note that they do not make any Noetherian hypothesis). There is a natural notion of morphism in this setting. The base extension is locally given by taking the semi-group ring, i.e. if A is a monoid with zero 0_A and $X = \mathrm{Spec}\, A$ is its spectrum, then

$$X_{\mathbb{Z}} \;:=\; X \otimes_{\mathbb{F}_1} \mathbb{Z} \;:=\; \mathrm{Spec}\big(\mathbb{Z}[A]/(1 \cdot 0_A - 0_{\mathbb{Z}[A]})\big).$$

An $\mathbb{F}_1$-scheme is a triple $(\tilde{X}, X, e_X)$, where $\tilde{X}$ is an M_0-scheme, X is a scheme and $e_X : \tilde{X}_\mathbb{Z} \to X$ is a morphism such that $e_X(k) : \tilde{X}_\mathbb{Z}(k) \xrightarrow{\sim} X(k)$ is a bijection for all fields k.

Note that an M_0-scheme $\tilde{X}$ defines the $\mathbb{F}_1$-scheme $(\tilde{X}, \tilde{X}_\mathbb{Z}, \mathrm{id}_{\tilde{X}_\mathbb{Z}})$. We give first examples of $\mathbb{F}_1$-schemes of this kind. The affine line $\mathbb{A}^1_{\mathbb{F}_1}$ is the spectrum of the monoid $\{T^i\}_{i\in\mathbb{N}} \amalg \{0\}$ and, indeed, we have $\mathbb{A}^1_{\mathbb{F}_1} \otimes_{\mathbb{F}_1} \mathbb{Z} \simeq \mathbb{A}^1$. The multiplicative group $\mathbb{G}_{m,\mathbb{F}_1}$ is the spectrum of the monoid $\{T^i\}_{i\in\mathbb{Z}} \amalg \{0\}$, which base extends to $\mathbb{G}_m$ as desired. Both examples can be extended to define $\mathbb{A}^n_{\mathbb{F}_1}$ and $\mathbb{G}^n_{m,\mathbb{F}_1}$ by considering multiple variables $T_1, \dots, T_n$. More generally, all $\mathbb{F}_1$-schemes in the sense of Deitmar deliver examples of M_0 and thus $\mathbb{F}_1$-schemes in this new sense. In particular, toric varieties can be realized.

To obtain a richer class of examples, we recall the definition of a torified variety as given in a joint work with Javier López Peña ([6]). A *torified variety* is a variety X together with morphism $e_X : T \to X$ such that $T \simeq \coprod_{i\in I} \mathbb{G}_m^{d_i}$, where I is a finite index set and d_i are non-negative integers and such that for every field k, the morphism e_X induces a bijection $T(k) \xrightarrow{\sim} X(k)$. We call $e_X : T \to X$ a *torification* of X.

Note that T is isomorphic to the base extension $\tilde{X}_\mathbb{Z}$ of the M_0-scheme $\tilde{X} = \coprod_{i\in I} \mathbb{G}^{d_i}_{m,\mathbb{F}_1}$. Thus every torified variety $e_X : T \to X$ defines an $\mathbb{F}_1$-scheme $(\tilde{X}, X, e_X)$.

In [6], a variety of examples are given. Most important for our purpose are toric varieties, Grassmannians and split reductive groups. If X is a toric variety of dimension n with fan $\Delta = \{\text{cones } \tau \subset \mathbb{R}^n\}$, i.e. $X = \mathrm{colim}_{\tau\in\Delta} \operatorname{Spec} \mathbb{Z}[A_\tau]$, where $A_\tau = \tau^\vee \cap \mathbb{Z}^n$ is the intersection the dual cone $\tau^\vee \subset \mathbb{R}^n$ with the dual lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Then the natural morphism $\coprod_{\tau\in\Delta} \operatorname{Spec} \mathbb{Z}[A_\tau^\times] \to X$ is a torification of X.

The Schubert cell decomposition of $\mathrm{Gr}(k, n)$ is a morphism $\coprod_{w\in M_{k,n}} \mathbb{A}^{d_w} \to \mathrm{Gr}(k, n)$ that induces a bijection of k-points for all fields k. Since the affine spaces in this decomposition can be further decomposed into tori, we obtain a torification $e_X : T \to \mathrm{Gr}(k, n)$. Note that the lowest-dimensional tori are 0-dimensional and the number of 0-dimensional tori is exactly $\#M_{k,n}$.

Let G be a split reductive group of rank r with maximal torus $T \simeq \mathbb{G}_m^r$, normalizer N and Weyl group $W = N(\mathbb{Z})/T(\mathbb{Z})$. Let B be a Borel subgroup containing T. The Bruhat decomposition $\coprod_{w\in W} BwB \to G$, where $BwB \simeq \mathbb{G}_m^r \times \mathbb{A}^{d_w}$ for some $d_w \geq 0$, yields a torification $e_G : T \to G$ analogously to the case of the Grassmannian. This defines a model $\mathcal{G} = (\tilde{G}, G, e_G)$ over $\mathbb{F}_1$. Note that in this case the lowest-dimensional tori are r-dimensional and that the number of r-dimensional tori is exactly $\#W$.

Clearly, there is a close connection between torified varieties and the $\mathbb{F}_1$-schemes in the sense of Connes and Consani with the idea that Tits had in mind. However, the natural choice of morphism in this category is a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ of M_0-schemes together with a morphism $f : X \to Y$ of schemes such that

$$\begin{array}{ccc} \tilde{X}_\mathbb{Z} & \xrightarrow{\tilde{f}_\mathbb{Z}} & \tilde{Y}_\mathbb{Z} \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_X} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Unfortunately, the only reductive groups G whose group law $m : G \times G \to G$ extends to a morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ in this sense such that $(\mathcal{G}, \mu)$ becomes a group object in the category of $\mathbb{F}_1$-schemes are algebraic groups of the form $G \simeq \mathbb{G}_m^r \times (\text{finite group})$. In the following section we will show how to modify the notion of morphism to realize Tits' idea.

4. Strong morphisms

Let $\mathcal{X} = (\tilde{X}, X, e_X)$ and $\mathcal{Y} = (\tilde{Y}, Y, e_Y)$ be $\mathbb{F}_1$-schemes. Then we define the *rank of a point* x in the underlying topological space $\tilde{X}$ as $\operatorname{rk} x := \operatorname{rk} \mathcal{O}^\times_{X,x}$, where $\mathcal{O}_{X,x}$ is the stalk (of monoids) at x and $\mathcal{O}_{X,x}^{\times}$ denotes its group of invertible elements. We define the *rank of* X as $\operatorname{rk} X := \min_{x\in\tilde{X}}\{\operatorname{rk} x\}$ and we let

$$\tilde{X}^{\mathrm{rk}} \;:=\; \coprod_{\operatorname{rk} x = \operatorname{rk}\tilde{X}} \operatorname{Spec}\mathcal{O}_{X,x}^{\times},$$

which is a sub-M_0-scheme of $\tilde{X}$. A *strong morphism* $\varphi : \mathcal{X} \to \mathcal{Y}$ is a pair $\varphi = (\tilde{f}, f)$, where $\tilde{f} : \tilde{X}^{\mathrm{rk}} \to \tilde{Y}^{\mathrm{rk}}$ is a morphism of M_0-schemes and $f : X \to Y$ is a morphism of schemes such that

$$\begin{array}{ccc} \tilde{X}^{\mathrm{rk}}_{\mathbb{Z}} & \xrightarrow{\;\tilde{f}_{\mathbb{Z}}\;} & \tilde{Y}^{\mathrm{rk}}_{\mathbb{Z}} \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_Y} \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

commutes.

This notion comes already close to achieving our goal. In the category of $\mathbb{F}_1$-schemes together with strong morphisms, the object $(\operatorname{Spec}\{0,1\}, \operatorname{Spec}\mathbb{Z}, \mathrm{id}_{\operatorname{Spec}\mathbb{Z}})$ is the terminal object, which we should define as $\operatorname{Spec}\mathbb{F}_1$. We define

$$\mathcal{X}(\mathbb{F}_1) \;:=\; \operatorname{Hom}_{\text{strong}}(\operatorname{Spec}\mathbb{F}_1, \mathcal{X}),$$

which equals the set of points of $\tilde{X}^{\mathrm{rk}}$ as every strong morphism $\operatorname{Spec}\mathbb{F}_1 \to \mathcal{X}$ is determined by the image of the unique point $\{0\}$ of $\operatorname{Spec}\{0,1\}$ in $\tilde{X}^{\mathrm{rk}}$. We see at once that $\#\mathcal{X}(\mathbb{F}_1) = \# M_{k,n}$ if $\mathcal{X}$ is a model of the Grassmannian $\operatorname{Gr}(k,n)$ as $\mathbb{F}_1$-scheme and that $\#\mathcal{G}(\mathbb{F}_1) = \# W$ if $\mathcal{G} = (\tilde{G}, G, e_G)$ is a model of a split reductive group G with Weyl group W.

Furthermore, if the Weyl group can be lifted to $N(\mathbb{Z})$ as group, i.e. if the short exact sequence of groups

$$1 \longrightarrow T(\mathbb{Z}) \longrightarrow N(\mathbb{Z}) \longrightarrow W \longrightarrow 1$$

splits, then from the commutativity of

$$\begin{array}{ccc} \tilde{G}^{\mathrm{rk}}_{\mathbb{Z}} \times \tilde{G}^{\mathrm{rk}}_{\mathbb{Z}} & \xrightarrow{\;\tilde{m}_{\mathbb{Z}}\;} & \tilde{G}^{\mathrm{rk}}_{\mathbb{Z}} \\ \downarrow{\scriptstyle (e_G, e_G)} & & \downarrow{\scriptstyle e_G} \\ G \times G & \xrightarrow{\;m\;} & G \end{array}$$

we obtain a morphism $\tilde{m} : \tilde{G}^{\mathrm{rk}} \times \tilde{G}^{\mathrm{rk}} \to \tilde{G}^{\mathrm{rk}}$ of M_0-schemes such that $\mu = (\tilde{m}, m) : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a strong morphism that makes $\mathcal{G}$ into a group object.

However, $\mathrm{SL}(n)$ provides an example where the Weyl group cannot be lifted. This leads us, in the following section, to introduce a second kind of morphisms.

5. Weak morphisms

The morphism $\operatorname{Spec}\mathcal{O}_{X,x}^{\times} \to *_{M_0}$ to the terminal object $*_{M_0} = \operatorname{Spec}\{0,1\}$ in the category of M_0-schemes induces a morphism

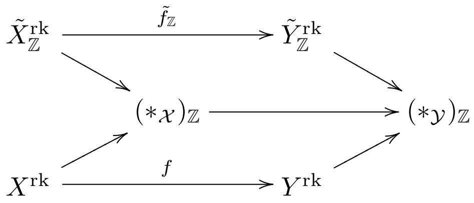
$$\tilde{X}^{\mathrm{rk}} \;=\; \coprod_{x\in\tilde{X}^{\mathrm{rk}}} \operatorname{Spec}\mathcal{O}_{X,x}^{\times} \;\longrightarrow\; *_{\mathcal{X}} \;:=\; \coprod_{x\in\tilde{X}^{\mathrm{rk}}} *_{M_0}.$$

Given $\tilde{f} : \tilde{X}^{\mathrm{rk}} \to \tilde{Y}^{\mathrm{rk}}$, there is a unique morphism $*_{\mathcal{X}} \to *_{\mathcal{Y}}$ such that

$$\begin{array}{ccc} \tilde{X}^{\mathrm{rk}} & \xrightarrow{\;\tilde{f}\;} & \tilde{Y}^{\mathrm{rk}} \\ \downarrow & & \downarrow \\ *_{\mathcal{X}} & \longrightarrow & *_{\mathcal{Y}} \end{array}$$

commutes. Let X^{rk} denote the image of $e_X : \tilde{X}^{\mathrm{rk}} \to X$. A *weak morphism* $\varphi : \mathcal{X} \to \mathcal{Y}$ is a pair $\varphi = (\tilde{f}, f)$, where $\tilde{f} : \tilde{X}^{\mathrm{rk}} \to \tilde{Y}^{\mathrm{rk}}$ is a morphism of M_0-schemes and $f : X \to Y$

is a morphism of schemes such that



commutes.

The key observation is that a weak morphism $\varphi = (\tilde{f}, f) : \mathcal{X} \to \mathcal{Y}$ has a base extension $f : X \to Y$ to $\mathbb{Z}$, but also induces a morphism $\tilde{f}_* : \mathcal{X}(\mathbb{F}_1) \to \mathcal{Y}(\mathbb{F}_1)$. With this in hand, we yield the following results.

6. Algebraic groups over $\mathbb{F}_1$

The idea of Tits' paper is now realized in the following form.

Theorem 6.1. *Let G be a split reductive group with group law $m : G \times G \to G$ and Weyl group W. Let $\mathcal{G} = (\tilde{G}, G, e_G)$ be the model of G as described before as $\mathbb{F}_1$-scheme. Then there is morphism $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ of M_0-schemes such that $\mu = (\tilde{m}, m)$ is a weak morphism that makes $\mathcal{G}$ into a group object. In particular, $\mathcal{G}(\mathbb{F}_1)$ inherits the structure of a group that is isomorphic to W.*

We have already seen that $\mathcal{X}(\mathbb{F}_1) = M_{k,n}$ when $\mathcal{X}$ is a model of $\mathrm{Gr}(n,k)$ as $\mathbb{F}_1$-scheme. Furthermore, we have the following.

Theorem 6.2. *Let $\mathcal{G}$ be a model of $G = \mathrm{GL}(n)$ as $\mathbb{F}_1$-scheme and let $\mathcal{X}$ be a model of $X = \mathrm{Gr}(k,n)$ as $\mathbb{F}_1$-scheme. Then the group action*

$$f: \ \mathrm{GL}(n) \times \mathrm{Gr}(k,n) \ \longrightarrow \ \mathrm{Gr}(k,n),$$

induced by the action on $\mathbb{P}^{n-1}$, can be extended to a strong morphism $\varphi : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ such that the group action

$$\varphi(\mathbb{F}_1): \ S_n \times M_{k,n} \ \longrightarrow \ M_{k,n},$$

of $\mathcal{G}(\mathbb{F}_1) = S_n$ on $\mathcal{X}(\mathbb{F}_1) = M_{k,n}$ is induced by the action on $M_n = \{1, \ldots, n\}$.

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