Towards a cohomological understanding of the tropical Riemann-Roch theorem

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Part 1: Tropical Riemann-Roch, aka Baker-Norine theory

Divisors on graphs

Let G be a graph with vertex set V. A **divisor** on G is an element of the abelian group

$$\operatorname{Div} G \;=\; \mathbb{Z}^V,$$

which we typically write as a formal linear combination $D = \sum D_v v$ of vertices $v \in V$ where $D_v = D(v)$.

A **principal divisor** in G is a divisor in the image of the group homomorphism

$$\operatorname{div}: \; \mathbb{Z}^V \;\longrightarrow\; \mathbb{Z}^V,$$

that sends an element $f \in \mathbb{Z}^V$ to the divisor $\operatorname{div}(f) = \sum df_v v$ with

$$df_v \;=\; \sum_{\text{edges } v - w} \big(f(w) - f(v)\big).$$

The rank

A divisor $D = \sum D_v v$ on G is **effective** is $D_v \geq 0$ for all $v \in V$.

We define $r_G(D) = 0$ if $D + \mathrm{div}(f)$ is *not* effective for any $f \in \mathbb{Z}^V$.

We define recursively

$$r_G(D) \; = \; 1 + \min \left\{ \, r_G(D - v) \, \middle| \, v \in V \, \right\}$$

if $D + \mathrm{div}(f)$ is effective for some $f \in \mathbb{Z}^V$.

The number $r_G(D)$ is called the **rank** of D.

Tropical Riemann-Roch

The **degree** of a divisor $D = \sum D_v v$ on G is $\deg(D) = \sum D_v$.

The **genus** of G is its first Betti number

$$g \;=\; h_1(G, \mathbb{Z}) \;=\; \#\{\text{edges}\} - \#V + 1.$$

The **canonical divisor** of G is the divisor $K = \sum K_v v$ on G with

$$K_v \;=\; \#\{\,\text{edges } v - w\,\} - 2.$$

Theorem (Baker-Norine 06)

Let D be a divisor on G. Then

$$r_G(D) - r_G(K - D) \;=\; \deg(D) + 1 - g.$$

Remark: Gathmann-Kerber and Mikhalkin-Zharkov have deduced from this theorem a version for metric graphs.

The specialization lemma

Let k be a field with discrete absolute value $v : k \to \mathbb{R}_{\geq 0}$, valuation ring $R \subsetneq k$ and residue field k_0.

Let X be a smooth projective curve over k and $\mathcal{X}$ a *strictly semistable R-model* of X, i.e. an R-scheme whose generic fibre $\mathcal{X}_k$ is isomorphic to X and whose special fibre $\mathcal{X}_0$ consists of transversally intersecting smooth curves over k_0.

Let G be the *dual graph* of $\mathcal{X}_0$ whose vertices correspond to the irreducible components of $\mathcal{X}_0$ and whose edges correspond to the nodes of $\mathcal{X}_0$.

Taking the Zariski closure in $\mathcal{X}$ of a divisor D of X induces a group homomorphism $\mathrm{trop} : \mathrm{Div}\, X \to \mathrm{Div}\, G$.

Theorem (Baker 07)

Let D be a divisor of rank $r_X(D)$ on X. Then $r_X(D) \leq r_G\big(\mathrm{trop}(D)\big)$.

Remark: This theory has applications in Brill-Noether theory.

Part 2: Towards a cohomological interpretation of the tropical Riemann-Roch theorem

Endowing G with a scheme structure

We recall the context:

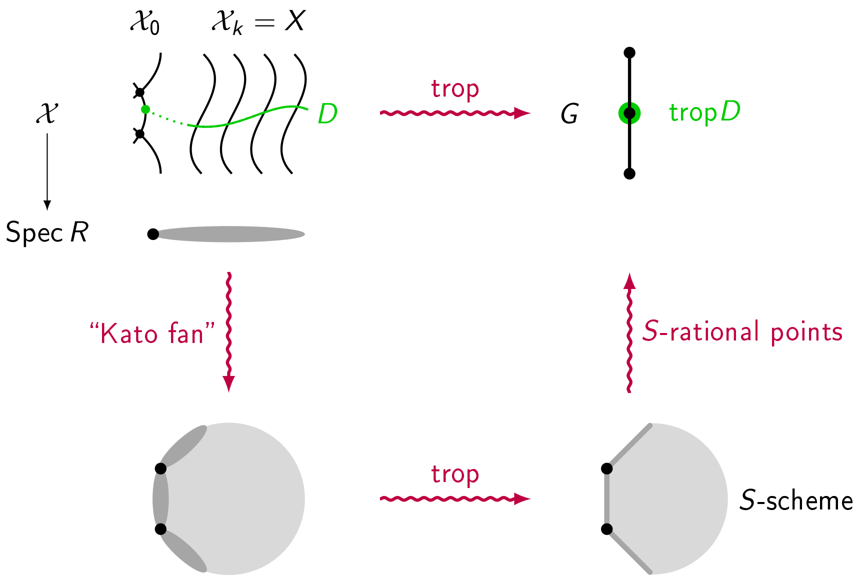
- a field k with discrete valuation $v : k \to \mathbb{R}_{\geq 0}$;
- its valuation ring R and residue field k_0;
- a smooth projective curve X over k;
- a strictly semistable R-model $\mathcal{X}$ of X;
- the dual graph G of $\mathcal{X}_0$.

We denote by $\mathbb{R}_{\geq 0}^{\max}$ the **tropical semifield** (using the Berkovich convention), which is $\mathbb{R}_{\geq 0}$ together with the usual multiplication and the addition

$$a + b = \max\{a, b\}.$$

The subsemiring $S = [0, 1]$ of $\mathbb{R}_{\geq 0}^{\max}$ is the **semiring of tropical integers**.

A picture



$\mathcal{X}_0$
$\mathcal{X}_k = X$
$\mathcal{X}$
D
Spec R
trop
G
trop D
"Kato fan"
S-rational points
trop
S-scheme

Remarks

- In joint work with Martin Ulirsch (in progress), we can make sense of the "Kato fan" and its tropicalization within the theory of *ordered blue schemes*.
- Sets of cocycles for tropical schemes (i.e. schemes over $\mathbb{R}_{\geq 0}^{\max}$) form naturally tropical linear spaces, aka *valuated matroids*. For S-schemes, we need a notion of *matroid bundles*.

Part 3: The tropical hyperfield

An intuitive definition

The **tropical hyperfield** is $\mathbb{R}_{\geq 0}$ together with the usual multiplication and the *hyperaddition*

$$a \boxplus b \;=\; \begin{cases} \{\max\{a,b\}\} & \text{if } a \neq b; \\ [0,a] & \text{if } a = b. \end{cases}$$

Note that

- $0 \boxplus a = a \boxplus 0 = \{a\}$ for all a (neutral element);
- $0 \in a \boxplus b$ if and only if $b = a$ (additive inverses);
- $c \in a \boxplus b$ if and only if the maximum occurs twice among a, b, c (tropical equality).

Part 4: Ordered blueprints

The definition

An **ordered blueprint** is a triple $B = (B^\bullet, B^+, \leq)$ where

- B^+ is a semiring (commutative with 0 and 1);
- $B^\bullet \subset B^+$ is a multiplicative subset that generates B^+ as a semiring and contains 0 and 1;
- $\leq$ is a partial order on B^+ that is *additive* and *multiplicative*, i.e. $x \leq y$ implies $x + z \leq y + z$ and $xz \leq yz$.

We call B^+ the **ambient semiring** and $B^\bullet$ the **underlying monoid** of B. We write $a \in B$ for $a \in B^\bullet$.

Given a subset S of $B^+ \times B^+$, we denote by $\langle S \rangle$ the smallest additive and multiplicative partial order on B^+.

A **morphism** of ordered blueprints is a multiplicative map $f : B_1 \to B_2$ that extends (necessarily uniquely) to an order-preserving semiring homomorphism $f^+ : B_1^+ \to B_2^+$.

This defines the category OBlpr of ordered blueprints. It is closed, complete and cocomplete and contains free objects.

Examples

- $\mathbb{F}_1 = \big(\{0,1\}, \mathbb{N}, =\big)$ (initial object)
- $\mathbb{B} = \big(\{0,1\}, (\{0,1\}, \max, \cdot), =\big)$ (Boolean semifield)
- $\mathbb{K} = \big(\{0,1\}, \mathbb{N}, \langle 0 \leq 1+1, 1 \leq 1+1 \rangle\big)$ (Krasner hyperfield)
- $\mathbb{T} = \big(\mathbb{R}_{\geq 0}, \mathbb{N}[\mathbb{R}_{>0}], \langle c \leq a+b \,|\, c \in a \boxplus b \rangle\big)$ (tropical hyperfield)

Fact

For $a_1, \dots, a_n, b \in \mathbb{T}$, we have $b \leq \sum a_i$ if and only if the maximum occurs twice among $a_1, \dots, a_n, b$.

- A field k defines the ordered blueprint $\mathbf{k} = \big(k, \mathbb{N}[k^\times], \langle c \leq a+b \,|\, c = a+b \text{ in } k \rangle\big)$.

Part 5: Scheme theory for ordered blueprints

Localizations

Let B be an ordered blueprint. The **unit group** of B is the group $B^\times$ of multiplicatively invertible elements of B.

Fact

Let $S \subset B$ be a multiplicative subset. Then there exists a universal morphism $\iota_S : B \to S^{-1}B$ among all morphisms $f : B \to C$ such that $f(S) \subset C^\times$, which we call the ***localization of B at S****.*

For $h \in B$, we define $B[h^{-1}] = S^{-1}B$ where $S = \{h^i\}_{i \in \mathbb{N}}$. We call $\iota_h = \iota_S : B \to B[h^{-1}]$ the **localization of B at h**.

Ordered blue schemes

Let $\mathrm{Aff} = \mathrm{OBlpr}^{\mathrm{op}}$. We denote the anti-equivalence of OBlpr with its opposite category by

$$\mathrm{Spec}: \ \mathrm{OBlpr} \ \longrightarrow \ \mathrm{Aff}\,.$$

Given a morphism $f : B \to C$ of ordered blueprints, we write $f^* : \mathrm{Spec}\, C \to \mathrm{Spec}\, B$ for the opposite morphism in Aff.

A **principal open immersion** is a morphism in Aff of the form $\iota_h^* : \mathrm{Spec}\, B[h^{-1}] \to \mathrm{Spec}\, B$.

Let $\mathcal{T}$ be the Grothendieck pretopology on Aff that is generated by families of principal open immersions $\big\{ \mathrm{Spec}\, B[h_i^{-1}] \to \mathrm{Spec}\, B \big\}_{i \in I}$ that are contained in the canonical topology of Aff.

An **ordered blue scheme** is the colimit of a "monodromy-free" diagram of principal open immersions in the category of sheaves $\mathrm{Sh}(\mathrm{Aff}, \mathcal{T})$ on the site $(\mathrm{Aff}, \mathcal{T})$.

Geometric points

Taking geometric points of the slice category over an ordered blue scheme X in $\mathrm{Sh}(\mathrm{Aff}, \mathcal{T})$ allows us to identify X with a topological space $\underline{X}$ together with a structure sheaf $\mathcal{O}_X$ in OBlpr.

In the case of an **affine ordered blue scheme** $X = \operatorname{Spec} B$, where we identify Aff with its essential image under the Yoneda embedding $\mathrm{Aff} \to \mathrm{Sh}(\mathrm{Aff}, \mathcal{T})$, the points of the underlying topological space of $\operatorname{Spec} B$ correspond to the *prime ideals* of $\operatorname{Spec} B$, which are subsets $\mathfrak{p}$ of B such that

- $0 \in \mathfrak{p}$,
- $\mathfrak{p} \cdot B = \mathfrak{p}$, and
- $S = B - \mathfrak{p}$ is a multiplicative subset.

Part 6: Tropicalization as a base change

Nonarchimedean absolute values as morphisms

Let k be a field. Recall the definitions of the associated ordered blueprint

$$\mathbf{k} \;=\; \Big(k,\, \mathbb{N}[k^\times],\, \langle c \leq a + b | c = a + b \text{ in } k \rangle\Big)$$

and the tropical hyperfield

$$\mathbb{T} \;=\; \Big(\mathbb{R}_{\geq 0},\, \mathbb{N}[\mathbb{R}_{>0}],\, \langle c \leq a + b | c \in a \boxplus b \rangle\Big).$$

Fact

A map $v : k \to \mathbb{R}_{\geq 0}$ is a nonarchimedean absolute value if and only if it is a morphism $\mathbf{v} : \mathbf{k} \to \mathbb{T}$ of ordered blueprints.

Blue models

Let $X = \operatorname{Spec} R$ be an affine k-scheme with a *choice of coordinates*, by which we mean a closed immersion $\iota : X \to \operatorname{Spec} k[A]$ into a toric variety where A is a suitable monoid (commutative, fine and saturated). Let $\pi : k[A] \to R$ the surjection of coordinate rings.

We associate with ι the ordered blue scheme $\mathbb{X} = \operatorname{Spec} B$ where

$$B \;=\; \Big(\big\{ ca \,\big|\, c \in k, a \in A \big\}, \, \mathbb{N}[k^\times \times A], \, \leq \Big)$$

whose partial order $\leq$ is generated by the relations $db \leq \sum c_i a_i$ for which $\pi(db) = \pi(\sum c_i a_i)$ in R.

The morphism $\mathbf{k} \to B$ that sends c to $c \cdot 1$ endows $\mathbb{X}$ with the structure of an ordered blue $\mathbf{k}$-scheme. We call $\mathbb{X}$ a **blue model** of X.

Tropicalization as a base change

The **scheme theoretic tropicalization** of X (with respect to ι) is the $\mathbb{T}$-scheme

$$\mathbb{X}^{\text{trop}} = \mathbb{X} \times_{\mathbf{k}} \mathbb{T} = \text{Spec}\left(B \otimes_{\mathbf{k}} \mathbb{T}\right)$$

where $B \otimes_{\mathbf{k}} \mathbb{T}$ is the colimit of $B \longleftarrow \mathbf{k} \xrightarrow{\mathbf{v}} \mathbb{T}$.

The **set theoretic tropicalization** of X (with respect to ι) is

$$X^{\text{trop}} = \left\{ f : A \to \mathbb{R}_{\geq 0} \,\middle|\, \underbrace{\begin{array}{l} \text{for all } \sum c_i a_i \in \ker\pi \text{ with } c_i \in k,\ a_i \in A, \\ \{v(c_i)f(a_i)\} \text{ assumes the maximum twice} \end{array}}_{\text{i.e. } 0 \leq \sum v(c_i) f(a_i) \text{ in } \mathbb{T}} \right\}$$

Theorem (L'19)

The composition with the natural map $A \to B \otimes_{\mathbf{k}} \mathbb{T}$, sending a to $(1 \cdot a) \otimes 1$, defines a bijection

$$\mathbb{X}^{\text{trop}}(\mathbb{T}) = \text{Hom}_{\mathbb{T}}(B \otimes_{\mathbf{k}} \mathbb{T}, \mathbb{T}) \longrightarrow X^{\text{trop}}.$$

Recovering the Giansiracusa tropicalization

In their seminal paper on tropical scheme theory, Jeff and Noah Giansiracusa introduce an $\mathbb{R}_{\geq 0}^{\max}$-scheme that represents X^{trop} in terms of the so-called **bend relation** on $\mathbb{R}_{\geq 0}^{\max}[A]$.

Theorem (L'19)

$\left(\mathbb{X}^{\text{trop}} \times_{\mathbb{F}_1} \mathbb{B}\right)^+ = \text{Spec}\left(B \otimes_{\mathbf{k}} \mathbb{T} \otimes_{\mathbb{F}_1} \mathbb{B}\right)^+$ *is naturally isomorphic to the Giansiracusa tropicalization of* X.

In particular, we recover the Giansiracusa bend relation on $\mathbb{R}_{\geq 0}^{\max}[A]$ as the congruence kernel of the projection

$$\mathbb{R}_{\geq 0}^{\max}[A] \longrightarrow \left(B \otimes_{\mathbf{k}} \mathbb{T} \otimes_{\mathbb{F}_1} \mathbb{B}\right)^+$$

induced by $\pi : k[A] \to R$.

Part 7: Matroid bundles
(joint work with Matthew Baker)

Idylls

The **regular partial field** is the ordered blueprint

$$\mathbb{F}_1^{\pm} \;=\; \Big(\{0,1,\epsilon\},\, \mathbb{N}[1,\epsilon],\, \langle 0 \leq 1+\epsilon\rangle\Big)$$

where $\epsilon^2 = 1$. Thus ϵ plays the role of an additive inverse of 1.

An **$\mathbb{F}_1^{\pm}$-algebra** is an ordered blueprint B together with a morphism $\mathbb{F}_1^{\pm} \to B$. By abuse of notation, we denote the image of ϵ in B also by ϵ.

An **idyll** is an $\mathbb{F}_1^{\pm}$-algebra $\mathbb{F}_1^{\pm} \to F$ such that

- $F^\bullet = \{0\} \cup F^\times$;
- $F^+ = \mathbb{N}[F^\times]$;
- for all relations of the form $0 \leq \sum c_i a_i$, we have either $\sum c_i \geq 3$ or $\sum c_i a_i = a + \epsilon a$ for some $a \in B$.

Remark: We interpret $\sum c_i a_i$ as zero if $0 \leq \sum c_i a_i$.

Examples

The regular partial field $\mathbb{F}_1^{\pm}$ is tautologically an idyll.

The Krasner hyperfield $\mathbb{K} = \big(\{0,1\}, \mathbb{N}, \langle 0 \leq 1+1, 1 \leq 1+1 \rangle\big)$ is an idyll with respect to the morphism $\mathbb{F}_1^{\pm} \to \mathbb{K}$ that sends ϵ to 1.

The tropical hyperfield $\mathbb{T}$ is an idyll with respect to the morphism $\mathbb{F}_1^{\pm} \to \mathbb{T}$ that maps ϵ to 1.

The ordered blueprint **k** associated with a field *k* is an idyll with respect to the morphism $\mathbb{F}_1^{\pm} \to \mathbf{k}$ that maps ϵ to -1.

Baker-Bowler theory

Fix $E = \{1, \ldots, n\}$ and $0 \leq r \leq n$. Let $\binom{E}{r} = \{r\text{-subsets of } E\}$.

Let F be an idyll. A **Grassmann-Plücker function in F** is a nontrivial function

$$\Delta : \binom{E}{r} \longrightarrow F$$

that satisfies the *Plücker relations*, i.e.

$$0 \leq \sum_{k=0}^{k=r} \epsilon^k \Delta\big(J - \{j_k\}\big)\Delta\big(J' \cup \{j_k\}\big)$$

for all $(r+1)$-subsets $J = \{j_0, \ldots, j_r\}$ and $(r-1)$-subsets J' of E where $j_0 < \cdots < j_r$ and $\Delta(J' \cup \{j_k\}) = 0$ if $j_k \in J'$.

An **F-matroid** is an $F^\times$-class $M = [\Delta]$ of a Grassmann-Plücker function $\Delta : \binom{E}{r} \to F$.

Theorem (Baker-Bowler 19)

Duality and cryptomorphisms (cycles, dual pairs) for F-matroids.

Examples

A **matroid** is a $\mathbb{K}$-matroid.

A **valuated matroid**, or **tropical linear space**, is a $\mathbb{T}$-matroid where $\mathbb{T}$ is the tropical hyperfield.

Let k be a field and $\mathbf{k}$ the associated idyll. Then a $\mathbf{k}$-matroid is a k-rational point of the Grassmannian $\mathrm{Gr}(r, E)$.

Matroids as rational points of a Grassmannian

Heuristic: An F-matroid should be an F-rational point of a Grassmannian "$\mathrm{Gr}(r, E)$". A *matroid bundle* on an ordered blue $\mathbb{F}_1^{\pm}$-scheme X should be a morphism "$X \to \mathrm{Gr}(r, E)$".

Certainly "$\mathrm{Gr}(r, E)$" cannot be a usual scheme. But we can turn this into a concise statement! Namely...

Let $B = (B^\bullet, B^+, \leq)$ be the $\mathbb{F}_1^{\pm}$-algebra with

- $B^+ = \mathbb{N}[1, \epsilon][T_I | I \in \binom{E}{r}]$ where $\epsilon^2 = 1$;
- $B^\bullet = \{c \cdot \prod T_I^{e_I} \big| c \in \{0, 1, \epsilon\}, e_i \in \mathbb{N}\}$;
- $\leq$ is generated by $0 \leq 1 + \epsilon$ and the Plücker relations.

The **matroid space** is defined as $\mathrm{Mat}(r, E) = \mathrm{Proj}\, B$ where Proj is defined analogously to usual algebraic geometry.

Theorem (Baker-L'18)

Let F be an idyll. Then $\mathrm{Mat}(r, E)(F)$ stays in a natural bijection with the set of all F-matroids $M = [\Delta : \binom{E}{r} \to F]$.

Digression: applications to matroid theory

Let F be an idyll. There is a unique morphism $t_F : F \to \mathbb{K}$. We say that a matroid M is **representable over F** if there is a Grassmann-Plücker function $\Delta : \binom{E}{r} \to F$ such that $M = [t_F \circ \Delta]$.

The morphism $\operatorname{Spec} \mathbb{K} \to \operatorname{Mat}(r, E)$ associated with a matroid M has a unique image point x_M. The **universal idyll** of M is the "residue field" k_M of $\operatorname{Mat}(r, E)$ at x_M.

Theorem (Baker-L'18)

Let F be an idyll and M a matroid. Then M is representable over F if and only if there is a morphism $k_M \to F$.

Remark: There is a derived object F_M, the **foundation** of M, that is better suited for applications than k_M. Representability is for many idylls F equivalent to the existence of a morphism $F_M \to F$. This leads to new proofs of various classical theorems on matroids.

Matroid bundles

Let X be an ordered blue $\mathbb{F}_1^{\pm}$-scheme. A **line bundle** on X is a sheaf $\mathcal{L}$ that is locally isomorphic to the structure sheaf $\mathcal{O}_X$.

A **Grassmann-Plücker function** on X is a line bundle $\mathcal{L}$ together with a function

$$\Delta : \binom{E}{r} \longrightarrow \Gamma(X, \mathcal{L})$$

such that $\left\{\Delta(I) | I \in \binom{E}{r}\right\}$ generates $\mathcal{L}$ and that satisfies the Plücker relations.

A **matroid bundle** on X is an isomorphism class $\mathcal{M} = [\Delta]$ of Grassmann-Plücker functions.

The moduli interpretation of matroid bundles

The matroid space comes with the usual Plücker embedding

$$\iota: \ \mathrm{Mat}(r,E) \ \longrightarrow \ \mathbb{P}^N_{\mathbb{F}_1^\pm} = \text{“}\operatorname{Proj}\mathbb{F}_1^\pm[T_I | I \in \tbinom{E}{r}]\text{”}$$

where $N = \#\binom{E}{r} - 1$. Let $\mathcal{O}(1)$ be the first twisted sheaf on $\mathbb{P}^N$. Then $T_I \in \Gamma(\mathbb{P}^N_{\mathbb{F}_1^\pm}, \mathcal{O}(1))$.

Theorem (Baker-L'18)

Let X be an ordered blue $\mathbb{F}_1^\pm$-scheme. Sending a morphism $\varphi: X \to \mathrm{Mat}(r,E)$ to the function $\Delta: \binom{E}{r} \to \Gamma(X, (\iota\circ\varphi)^*(\mathcal{O}(1))$ with $\Delta(I) = (\iota\circ\varphi)^\#(T_I)$ defines a bijection

$$\mathrm{Hom}_{\mathbb{F}_1^\pm}\big(X, \mathrm{Mat}(r,E)\big) \ \longrightarrow \ \big\{\textit{matroid bundles on } X\big\}.$$

In other words, $\mathrm{Mat}(r,E)$ is the fine moduli space of matroid bundles on X.

Matroids over the tropical integers

The **hyperring of tropical integers** is the ordered blueprint

$$\mathcal{O}_{\mathbb{T}} \;=\; \Big([0,1],\, \mathbb{N}\big[(0,1]\big],\, \langle c \leq a+b \,|\, c \leq a+b \text{ in } \mathbb{T}\rangle\Big),$$

which is an $\mathbb{F}_1^{\pm}$-algebra with respect to the morphism $\mathbb{F}_1^{\pm} \to \mathcal{O}_{\mathbb{T}}$ that sends ϵ to 1.

Since $\mathcal{O}_{\mathbb{T}}^{\times} = \{1\}$ and all line bundles on $\operatorname{Spec} \mathcal{O}_{\mathbb{T}}$ are trivial, a matroid bundle on $\operatorname{Spec} \mathcal{O}_{\mathbb{T}}$ is the same as a function

$$\Delta : \; \binom{E}{r} \;\longrightarrow\; \mathcal{O}_{\mathbb{T}}$$

such that $\Delta(I) = 1$ for some $I \in \binom{E}{r}$ and that satisfies the Plücker relations.

Part 8: Back to the tropical Riemann-Roch theorem

The next accomplished steps

Recall the context from the beginning: a field k with discrete valuation $v : k \to \mathbb{R}_{\geq 0}$ and valuation ring R; a smooth projective k-curve X with strictly semistable R-model $\mathcal{X}$.

The Kato fan of $\mathcal{X}$ provides the "coordinates" for a blue model $\mathbb{X}$ of $\mathcal{X}$. Its tropicalization is $\mathbb{X}^{\text{trop}} = \mathbb{X} \times_{\mathbf{R}} \mathcal{O}_{\mathbb{T}}$ where $\mathbf{R}$ is the ordered blueprint associated with R.

Fact

The (geometric realization of the) dual graph G of $\mathcal{X}_0$ stays in natural bijection with $\mathbb{X}^{\text{trop}}(\mathcal{O}_{\mathbb{T}})$.

Future steps

- Do cocycle sets for line bundles on $\mathbb{X}^{\mathrm{trop}}$ carry the expected matroid structure? Note that this requires a cryptomorphic description of $\mathcal{O}_{\mathbb{T}}$-matroids in terms of cycles (i.e. defining equations).
- Is the rank of the cohomology group equal to the Baker-Norine rank? Or does it give a sharper bound in the specialization lemma? (This could be very useful for Brill-Noether theory)
- Can we find a cohomological proof for the tropical Riemann-Roch theorem?
- What about Riemann-Roch for other ordered blueprints, e.g. $\mathbb{F}_1^{\pm}$?