Exercises for Algebra II Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 17.8.2015 Roberto Alvarenga Jr. (monitor)

Exercise 1.
Let L/K be a field extension and $a \in L$ algebraic over K. Let $f(T) \in K[T]$ be the minimal polynomial of a over K. Show that the minimal polynomial of the K-linear map

$$M_a: \begin{array}{ccc} L & \longrightarrow & L \\ b & \longmapsto & a \cdot b \end{array}$$

is equal to f.

Exercise 2.
Let L/K be a finite field extension. Then there are elements $a_1, \ldots, a_n \in L$ such that $L = K(a_1, \ldots, a_n)$.

Exercise 3.
Let L/K be a field extension and $a_1, \ldots, a_n \in L$. Show that $K(a_1, \ldots, a_n)/K$ is algebraic if and only if $a_1, \ldots, a_n$ are algebraic over K.

Exercise 4.
Consider the following elements $\sqrt[3]{2}$ and ζ_3 as elements of an algebraic closure of $\mathbb{Q}$.

1. Show that $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$ and find its minimal polynomial. What is the degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$?
2. Let $\zeta_3 = e^{2\pi i/3}$ be a *primitive third root of unity*, i.e. an element $\neq 1$ that satisfies $\zeta_3^3 = 1$. Show that ζ_3 is algebraic over $\mathbb{Q}$ and find its minimal polynomial. What is the degree $[\mathbb{Q}(\zeta_3) : \mathbb{Q}]$?
3. What is the degree of $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ over $\mathbb{Q}$?