Exercises for Algebra II — Instituto Nacional de Matemática Pura e Aplicada
Series 4 — Carolina Araujo and Oliver Lorscheid
To hand in until 31.8. — Roberto Alvarenga Jr. (monitor)

Exercise 1.
Calculate the Galois groups of the splitting fields of the following polynomials over $\mathbb{Q}$.

1. $f_1 = T^3 - 1$;
2. $f_2 = T^3 - 2$;
3. $f_3 = T^3 + T^2 - 2T - 1$.

Hint: If ζ is a 7-th root of 1 (different from 1), then $\zeta^i + \zeta^{7-i}$ is a root of f_3 for $i = 1, 2, 3$.

Exercise 2.
Show that $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$ is Galois. What is the Galois group? Make a diagram of all subgroups of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ that illustrates which subgroups are contained in others. Use this to determine all intermediate extensions of $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$.

Exercise 3.
Let K be a field and G a finite subgroup of the multiplicative group $K^\times$. Show that G is cyclic, which can be done along the following lines.

1. Let $\varphi(d)$ be the number of generators of a cyclic group of order d. Show for $n \geq 1$ that
$$\sum_{d|n} \varphi(d) \quad = \quad n.$$
Remark: The function $\varphi(d)$ is called *Euler's φ-function.*
2. Let $G_d \subset G$ be the subset of elements of order d. Show that G_d is empty if d is not a divisor of n and that G_d has exactly $\varphi(d)$ elements if it is not empty.
Hint: Use that $T^d - 1$ has at most d roots in a field.
3. Let n be the cardinality of G. Conclude that G must have an element of order n and that G is cyclic.

Exercise 4.

1. Find a finite separable (but not normal) field extension L/K that does not satisfy the Galois correspondence.
2. Find a finite normal (but not separable) field extension L/K that does not satisfy the Galois correspondence.
3. **(Bonus)** Find a normal and separable (but not finite) field extension L/K that does not satisfy the Galois correspondence.