
Exercise 1. Let L/K be a Galois extension and let

$$M_a : \begin{array}{ccc} L & \longrightarrow & L \\ b & \longmapsto & a \cdot b \end{array}$$

be the K-linear map associated with an element $a \in L$. Show that the trace of M_a equals $\mathrm{Tr}_{L/K}(a)$ and that the determinant of M_a equals $\mathrm{N}_{L/K}(a)$.

Hint: Use Exercise 1 from List 2.

Exercise 2. Let L be the splitting field of $T^3 - 2$ over $\mathbb{Q}$. Show that $\sqrt[3]{2}$, $\sqrt{-3}$ and ζ_3 are elements of L. Calculate $\mathrm{N}_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$ for $a = \sqrt[3]{2}$, $a = \sqrt{-3}$ and $a = \zeta_3$. Calculate $\mathrm{N}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$.

Exercise 3.
Let L be the splitting field of $f = T^4 - 3$ over $\mathbb{Q}$. What is the Galois group of $L/\mathbb{Q}$? Make a diagram of all subgroups of $\mathrm{Gal}(L/K)$ that illustrates which subgroups are contained in others. Which of the subextensions of $L/\mathbb{Q}$ are elementary radical? Is $L/\mathbb{Q}$ radical?

Hint: Find the four complex roots $a_1, \ldots, a_4$ of f. Which permutations of $a_1, \ldots, a_4$ extend to field automorphisms of L?

Exercise 4.
Let K be a field and L the splitting field of a polynomial f over K of degree 4. Show that L/K is solvable if it is separable.