Exercises for Algebraic Number Theory Oliver Lorscheid
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This list is not to hand in!

***Exercise 1.**
Show that

$$\begin{array}{ccc} \mathbb{Z}[[T]] & \longrightarrow & \mathbb{Z}_p \\ \sum_{i\geq 0} a_i T^i & \longmapsto & \sum_{i\geq 0} a_i p^i \end{array}$$

with $a_i \in \{0, \dots, p-1\}$ is a well-defined homomorphism of rings that is surjective and has kernel $(T-p)$.

***Exercise 2.**
The p-adic integer $\sum_{i\geq 0} a_i p^i$ is in $\mathbb{Z}_p^\times$ if and only if $a_0 \neq 0$.

***Exercise 3.**
$\mathbb{Q}_p$ and $\mathbb{Q}_q$ are isomorphic (as fields) for two prime numbers p and q if and only if $p = q$.
Hint: Make use of the different splitting behaviours of polynomials in $\mathbb{Z}[T]$ modulo p and q, respectively.

***Exercise 4.**
Determine all places of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{5})$.

***Exercise 5.**
Determine all archimedean places of $\mathbb{Q}(\zeta_n)$ and $\mathbb{Q}(\sqrt[n]{2})$ where $n \geq 1$ and ζ_n is a primitive n-th root of unity.

***Exercise 6.**
Let L/K be a finite separable extension, v a discrete valuation of K with valuation ring $A = \mathcal{O}_v$ and B the integral closure of A in L. Let $\hat{v}$ be the extension of v to the completion K_v of K at v, and $\mathcal{O}_{\hat{v}}$ its valuation ring. Show that

$$B \otimes_A \mathcal{O}_{\hat{v}} \;=\; \prod_{w|v} \mathcal{O}_{\hat{w}}$$

where $w|v$ varies over all extensions w of v to L and $\mathcal{O}_{\hat{w}}$ is the valuation ring of the extension $\hat{w}$ of w to the completion L_w of L at w.

***Exercise 7.**
What is the relation of Proposition 1 in section 2.6 to Proposition 1 in section 1.7 of the lecture?

***Exercise 8.**
Let K be a field of positive characteristic p.

1. Show that $K_0((T))/K_0(T)$ is not an algebraic extension.
2. Let $X \in K_0((T))$ be transcendental over $K_0(T)$, $K = K_0(T, X^p)$ and $L = K_0(T, X)$. Show that L/K has degree p and that the fundamental identity does not hold for the T-adic valuation v_T of K.

Hint: Use that v_T extends uniquely to $K_0((T))$.

The starred exercises are not to hand in.