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**Exercise 1.**
Derive the weak form of Hilbert's Nullstellensatz from the strong form.

**Exercise 2.**
Find all singular points of the plane affine curves

$$C_1 = V(T_1T_2-1), \quad C_2 = V(T_1^2-T_1T_2+T_2^2), \quad C_3 = V(T_1^3+2T_1^2+T_1-T_2^2-2T_2-1).$$

**Exercise 3.**
Let $C = V(f)$ be a plane affine curve in $\mathbb{A}^2_{\mathbb{C}}$. A subset $S$ of $C$ is called closed if there is a $g \in \mathbb{C}[T_1, T_2]$ such that $S = C \cap V(g)$.

1. Show that the closed subsets of $C$ form a topology of closed subsets for $C$.
2. Show that a curve $C$ in $\mathbb{A}^2_{\mathbb{C}}$ is irreducible if and only if it cannot be covered by two proper closed subsets.
3. In case that $C$ is irreducible, show that a subset $S$ of $C$ is closed if and only if $S = C$ or $S$ is finite.

**Exercise 4.**
Prove Hilbert's Basissatz and Krull's principal ideal theorem.

**Exercise 5.**
Let $A$ be a ring and $I$ an ideal of $A$. Show that $I$ is radical if and only if the only nilpotent element of $A/I$ is 0, i.e. $a^n = 0$ for $n > 0$ implies $a = 0$ for $a \in A/I$.

**Exercise 6.**
Let $A$ be a ring. Show that any two elements $a, b \in A$ form an $A$-linear dependent set $\{a, b\}$. Conclude that every free $A$-submodules of $A$ must be of rank 1. Use this to show that if every submodule of a free $A$-module is free, then $A$ is a principal ideal domain.

**Exercise 7.**
Let $A$ be a principal ideal domain and $M$ a free $A$-module with basis $\{b_1, \dots, b_n\}$. For an element $m = \sum m_i.b_i$, define the ideal $I(m) = (m_1, \dots, m_n)$ of $A$. Show that $I(m)$ does not depend on the choice of basis of $M$, and that $m$ occurs as an element of a basis for $M$ if and only if $I(m) = A$.

**Exercise 8.**
Let $A$ be a principal ideal domain and

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

be a short exact sequence of finitely generated $A$-modules.

1. Show that $M = 0$ if and only if $N = P = 0$.
2. Show that $\mathrm{rk}M = \mathrm{rk}N + \mathrm{rk}P$.

**Exercise 9.**
Consider the group homomorphism $\varphi : \mathbb{Z}^4 \to \mathbb{Z}^4$ given by the matrix

$$\begin{pmatrix} 0 & 5 & -2 & -1 \\ 3 & 1 & 1 & 3 \\ 2 & -4 & 4 & 2 \\ 8 & 8 & -8 & 8 \end{pmatrix}$$

Determine the Smith normal form of $\varphi$ and the decomposition of $\mathbb{Z}^4/\mathrm{im}\varphi$ into cyclic factors. What are the invariant factors and the elementary divisors of $\mathbb{Z}^4/\mathrm{im}\varphi$?

**Exercise 10.**
Let $M = \mathbb{C}^4$ be the $\mathbb{C}[T]$-module where $T$ acts as one of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Determine the minimal polynomial and the characteristic polynomial of each matrix, and the invariant factors and elementary divisors for each module structure on $M$.

**Exercise 11.**
Let $G$ be an abelian group with 32 elements. Show that $G$ is cyclic if and only if $G$ has no subgroup $H$ of order 2 or 4 such that

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

is split.

**Exercise 12.**
Let $A$ be a ring and $M$ an $A$-module. Show that

$$0 \longrightarrow N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0$$

is exact if $0 \to N_1 \to N_2 \to N_3 \to 0$ is a split exact sequence.

**Exercise 13.**
Let $A$ be a ring and $I$ and $J$ ideals of $A$. Show that $(A/I) \otimes_A (A/J)$ is isomorphic to $A/(I+J)$.

**Exercise 14.**
Let $A$ be a ring and $f : A^n \to A^n$ be the $A$-linear map that is defined by the matrix $(a_{i,j})$. Show that $\Lambda^n(f) : \Lambda^n A^n \to \Lambda^n A^n$ is multiplication by $\det(a_{i,j})$ under the identification $\Lambda^n A^n$ with $A$ given by the standard basis of $A^n$.

**Exercise 15.**
Let $k$ be a field, $V$ a finite dimensional $k$-vector space and $V^* = \mathrm{Hom}_k(V,k)$ the dual dual space. Describe an isomorphism $\varphi_l : \Lambda^l(V^*) \to (\Lambda^l V)^*$ for every $l \geq 0$.

**Exercise 16.**
Let $A$ be a ring and $M$ an $A$-module. A *free resolution of* $M$ is an exact sequence of the form

$$\longrightarrow \quad N_i \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad N_0 \quad \longrightarrow \quad M \quad \longrightarrow \quad 0$$

where $N_i$ is a free $A$-module for all $i \geq 0$. The *length* of the free resolution is the supremum over all indices $i$ such that $N_i \neq 0$.

1. Show that every $A$-module has a free resolution.
2. Let $A = k[T]$ be a polynomial ring over a field $k$. Show that every $A$-module $M$ with $\dim_k M < \infty$ has a free resolution of length 1.
3. Show that $M = k$ as the $k[T_1, T_2]$-module with $T_1.m = 0 = T_2.m$ for $m \in M$ does not have a free resolution of length 1.