Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 4 Oliver Lorscheid
To hand in at 13.4. in the exercise class Esteban Arreaga (monitor)

Exercise 1.
Let A be an integral domain and (a) a non-zero principal ideal of A. A *factorization of* (a) *into principal prime ideals* is an equality of the form $(a) = \prod_{i=1}^{n}(p_i)$ where (p_i) are principal prime ideals of A.

1. Show that a factorization in principal prime ideals is unique, i.e. if $(a) = \prod_{i=1}^{n}(p_i)$ and $(a) = \prod_{j=1}^{m}(q_j)$ are two such factorizations, then there exists a bijection $\sigma : \{1,\dots,n\} \to \{1,\dots,m\}$ such that $(p_i) = (q_{\sigma(i)})$ for all $i = 1,\dots,n$.

2. Show that A is a unique factorization domain if and only if every principal ideal of A has a factorization into principal prime ideals.

Exercise 2.
Let A be a unique factorization domain.

1. Show that every prime ideal of A is generated by a set of prime elements.

2. Find an example of a unique factorization domain A and prime elements $p_1,\dots,p_n$ of A such that $I = (p_1,\dots,p_n)$ is **not** a prime ideal.

3. Show that the ideal $I = (2, 1+\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$ is prime and that it does not contain any prime element.

Exercise 3.
Let A be a ring and $S \subset A$ a multiplicative subset.

1. Show that

$$\frac{s}{r}\cdot\frac{r}{s} = \frac{1}{1}, \qquad \frac{ta}{ts} = \frac{a}{s}, \qquad \text{and} \qquad \frac{a}{s}+\frac{b}{s} = \frac{a+b}{s}$$

   for all $a, b \in A$ and $r, s, t \in S$.

2. Show that $\frac{a}{s} = \frac{a'}{s'}$ if and only if $sa' = s'a$, in case that A is an integral domain.

3. Show that $S^{-1}A = \{0\}$ if $0 \in S$.

4. Show that $\iota_S : A \to S^{-1}A$ is injective if A is an integral domain and $0 \notin S$.

5. Let $A = A_1 \times A_2$ and $h = (1,0)$. Show that the association $\frac{(a,b)}{h^i} \mapsto (a,0)$ defines a ring isomorphism $A[h^{-1}] \simeq A_1$.

Exercise 4.
Let A be a non-zero ring, in which every proper ideal is prime. Show that A is a field.

Exercise 5 (Bonus exercise)**.**
Let A be a ring. The *spectrum of* A is the set $\mathrm{Spec}A$ of all prime ideals of A. A *principal open subset of* $\mathrm{Spec}A$ is a subset of the form

$$U_a \;=\; U_{A,a} \;=\; \left\{\, \mathfrak{p} \in \mathrm{Spec}A \,\middle|\, a \notin \mathfrak{p} \,\right\}$$

with $a \in A$.

1. Show that $U_0 = \emptyset$, $U_1 = \mathrm{Spec}A$ and $U_a \cap U_b = U_{ab}$ for all $a, b \in A$.
   Remark: This shows that the principal open subsets of $\mathrm{Spec}A$ form a basis for a topology on $\mathrm{Spec}A$, which is called the *Zariski topology.*

2. Let $f : A \to B$ be a ring homomorphism. By Exercise 2 of List 2, the association $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ defines a map $\varphi : \mathrm{Spec}B \to \mathrm{Spec}A$. Show that $\varphi^{-1}(U_{A,a}) = U_{B,f(a)}$ for every $a \in A$.
   Remark: This shows that the map $\varphi : \mathrm{Spec}B \to \mathrm{Spec}A$ is a continuous map.

3. Let $h \in A$ be a multiplicative subset and $f : A \to A[h^{-1}]$ the localization map. Show that $\varphi : \mathrm{Spec}A[h^{-1}] \to \mathrm{Spec}A$ is injective and satisfies $\varphi(U_{A[h^{-1}],\frac{a}{s}}) = U_{A,ah}$ for every $a \in A$ and $s = h^i$ with $i \geq 0$.
   Remark: This shows that $\varphi : \mathrm{Spec}(A[h^{-1}]) \to \mathrm{Spec}A$ is an open topological embedding with image U_h.

4. Let $V_{A,a} = \{\mathfrak{p} \in \mathrm{Spec}A | a \in \mathfrak{p}\}$ be the complement of $U_{A,a}$ in $\mathrm{Spec}A$. Let I be an ideal of A, $f : A \to A/I$ the canonical projection and $V(I) = \bigcap_{a \in I} V_{A,a}$. Show that $\varphi : \mathrm{Spec}(A/I) \to \mathrm{Spec}A$ is injective and satisfies $\varphi(V_{A/I,\overline{a}}) = V_{A,a} \cap V(I)$ for every $a \in A$ with residue class $\overline{a}$ in A/I.
   Remark: This shows that $\varphi : \mathrm{Spec}(A/I) \to \mathrm{Spec}A$ is a closed topological embedding with image $V(I)$.