Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
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To hand in at 20.4. in the exercise class Esteban Arreaga (monitor)

Exercise 1.
Let A be a ring and S a multiplicative subset. Show the following assertions.

1. The localization map $A \to S^{-1}A$ is injective if and only if for every $a \in S$, the multiplication $m_a : A \to A$ by a is an injective map.
2. If A is an integral domain, a unique factorization domain, a principal ideal domain, an Euclidean ring or a field and $0 \notin S$, then $S^{-1}A$ is so, too.
3. Find an example of a local ring A and a multiplicative subset S with $0 \notin S$ such that $S^{-1}A$ is not local.

Exercise 2.
Let A be a ring.

1. Show that $A[T_1, T_2] \simeq \big(A[T_1]\big)[T_2]$.
2. Let $f \in A[T]$ be nonzero. Show that $A[T][f^{-1}] \simeq A[T, T']/(fT' - 1)$.

Exercise 3.
Let K be a field and $f \in K[T]$ a polynomial.

1. Show for $\deg f = 2$ and $\deg f = 3$ that f is irreducible in $K[T]$ if and only if f does not have a root in K.
2. Find a field K and a polynomial $f \in K[T]$ of degree 4 that is not irreducible and does not have a root in K.
3. If f is irreducible of degree n, then $L = K[T]/(f)$ is $\dim_K L = n$.
4. Show that there exists a field extension L/K such that f factorizes in $L[T]$ as
$$f = u \prod_{i=1}^{n} (T - a_i)$$
with $u, a_1, \dots, a_n \in L$.

Exercise 4.
Let G be an abelian group with n elements. We define the *exponent of* G as the smallest positive integer m such that $g^m = e$ for all $g \in G$.

1. Show that G is cyclic if and only if its exponent is n.
2. Let K be a field and U a finite subgroup of order n of the multiplicative group $K^\times$ of K. Show that U is cyclic.
Hint: If m is the exponent of U, then every element of U is a zero of $T^m - 1$.

Exercise 5 (Bonus exercise)**.**

1. Show that all irreducible polynomials in $\mathbb{R}[T]$ are of degree 1 or 2.

2. Define two complex numbers z and z' as equivalent if $z' = z$ or $z' = \overline{z}$, the complex conjugate of z. Denote the corresponding equivalence relation by $\sim$ and the class of z in the quotient set $\mathbb{C}/\sim$ by $[z]$. Show that the map

$$\begin{array}{ccc} \mathbb{C}/\sim & \longrightarrow & \{\text{maximal ideals of } \mathbb{R}[T]\} \\ [z] & \longmapsto & \Big(\prod_{z' \in [z]} (T - z')\Big) \end{array}$$

   is a bijection.

3. Make a drawing of $\mathrm{Spec}\mathbb{R}[T]$ and of the map $f^* : \mathrm{Spec}\mathbb{C}[T] \to \mathrm{Spec}\mathbb{R}[T]$ that is induced by the inclusion $f : \mathbb{R}[T] \to \mathbb{C}[T]$.