Exercises for Commutative Algebra
List 0 not to hand in

Exercise 1.
Let A be a ring and M an abelian group. Let $\mathrm{End}(M)$ be the endomorphism ring of M (which is, in general, non-commutative). Show that the choice of a scalar multiplication $A \times M \to M$ that turns M into an A-module is the same as a ring homomorphism $A \to \mathrm{End}(M)$. In other words, an A-module is the same as an abelian group M together with a ring homomorphism $A \to \mathrm{End}(M)$.

Exercise 2.
Let A be a ring. Verify that the following constructions yield A-modules:

1. The homomorphism set $\mathrm{Hom}_A(M, N)$ for A-modules M and N.
2. The submodule $\langle S \rangle$ of an A-module M that is generated by a subset $S \subset M$.
3. The quotient of an A-module by a submodule $N \subset M$.
4. The intersection $\bigcap N_i$ of a family $\{N_i\}_{i \in I}$ of submodules of an A-module M.
5. The image, kernel and cokernel of a homomorphism $f : M \to N$ of A-modules.
6. The direct sum and the product of a family $\{M_i\}_{i \in I}$ of A-modules.
7. Let $f : A \to B$ be a ring homomorphism and M a B-module. Then M is an A-module via $a.m = f(a).m$.

Exercise 3.
Show that every abelian group has a unique structure as a $\mathbb{Z}$-module.

Exercise 4.
Let k be a field. Show that a k-module is the same as a k-vectorspace.

Exercise 5.
Let A be a ring, M an A-module and $N \subset M$ a submodule. Show that the restrictions of the addition and scalar multiplication of M to N turn N into an A-module. Show that the inclusion $N \hookrightarrow M$ is A-linear. Conclude that there is a bijection

$$\{ \text{ submodules } N \text{ of } M \ \} \quad \longrightarrow \quad \{ \text{ injective } A\text{-linear maps } N \to M \ \} / \simeq$$

for a fixed A-module M where we declare two injective A-linear maps $f : N \to M$ and $f' : N' \to M$ as isomorphic ("$\simeq$") if there is an isomorphism $g : N \to N'$ of A-modules such that $f = f' \circ g$.

Exercise 6.
Let A be a ring, M an A-module and $N \subset M$ a submodule.

1. Show that M/N is an A-module and that $\pi : M \to M/N$ is a homomorphism of A-modules.

2. Show that π satisfies the following universal property: for all homomorphism $f : M \to P$ such that $f(N) = \{0\}$, there is a unique homomorphism $\bar{f} : M/N \to P$ such that $f = \bar{f} \circ \pi$.

3. Conclude that every other surjective A-linear map $\pi' : M \to Q$ with $N = (\pi')^{-1}(0)$ is uniquely isomorphic ("$\simeq$") to π, i.e. there is a unique isomorphism $g : M/N \to Q$ such that $\pi' = g \circ \pi$.

4. Show that there is a bijection
$$\big\{ \text{ submodules } N \text{ of } M \,\big\} \quad \longrightarrow \quad \big\{ \text{ surjective } A\text{-linear maps } M \to Q \,\big\} / \simeq$$
for a fixed ring M.

5. Show that the association $N \mapsto \pi(N)$ defines a bijection
$$\big\{ \text{ submodules } P \text{ of } M \text{ with } N \subset P \,\big\} \quad \longrightarrow \quad \big\{ \text{ submodules of } M/N \,\big\}.$$

6. Show that for a subset P of M the following are equivalent.
   a) P is a submodule of M.
   b) There is an A-linear map $f : M \to Q$ such that $P = f^{-1}(0)$.
   c) $P = \pi^{-1}(0)$ for the canonical projection $\pi : A \to A/\langle P \rangle_A$.

Exercise 7.
Let A be a ring and M be an A-module. A *basis for* M is a family $\{\beta_i\}_{i \in I}$ of elements $\beta_i \in M$ such that for every $m \in M$, there is a unique element $(a_i) \in \bigoplus_{i \in I} A$ such that $m = \sum a_i.\beta_i$.

1. Show that M is free if and only if M has a basis.

2. Show that a free module M together with a basis $\{\beta_i\}_{i \in I}$ satisfies the following universal property: every map $f : \{b_i\} \to N$ into an A-module N extends uniquely to an A-linear map $M \to N$.

3. Show that the cardinalities of different bases for M are equal.

4. Show that $\prod_{i \in I} A$ is not free if I is infinite.

Exercise 8.
Prove the three isomorphism theorems for A-modules.

Exercise 9. Let A be a ring.

1. Show that $0 = 1$ if and only if $A = \{0\}$.
2. Show that a ring A is an integral domain if and only if for every nonzero $a \in A$, the multiplication $m_a : A \to A$ by a, defined by $m_a(b) = ab$, is injective.
3. Show that the following are equivalent.
   a) A is a field.
   b) The map m_a is bijective for every nonzero $a \in A$.
   c) The only ideals of A are (0) and (1).
   d) Every ring homomorphism $f : A \to B$ to a nonzero ring B is injective.

Exercise 10.
Let A be a ring and $B \subset A$ a subring. Show that $0 \in A$ and that the restrictions of the addition and multiplication of A to B turn B into a ring. Show that the inclusion $B \hookrightarrow A$ is a ring homomorphism.

Exercise 11.
Let A be a ring and I an ideal of A. Show that A/I is a ring and that $\pi : A \to A/I$ is a ring homomorphism.

Exercise 12.
Let A be a ring and I an ideal of A. Show that I is a prime ideal if and only if A/I is an integral domain, and that I is a maximal ideal if and only if A/I is a field.

Exercise 13.
Let A be a principal ideal domain and $a \in A$ be nonzero. Then (a) is a maximal ideal.

Exercise 14.
Show that every Euclidean domain is a principal ideal domain.

Exercise 15.
Let A be a ring.

1. Show that $A^\times$ acts on A by multiplication, i.e. $1.c = c$ and $(ab).c = a.(b.c)$ for all $a, b \in A^\times$ and $c \in A$.
2. Show that the association $[a] \mapsto (a)$ yields a (well-defined) bijection

$$A/A^\times \quad \longrightarrow \quad \{ \text{ principal ideals of } A \ \}.$$

Exercise 16.
Let A be a ring. Show that every ideal $I \neq (1)$ of A is contained in a maximal ideal. Conclude that if $a \in A - A^\times$, then a is contained in a maximal ideal.

Exercise 17.
Let A be a ring and $\mathfrak{m}$ a maximal ideal of A. The the following are equivalent.

1. A is a local ring.
2. $A = A^\times \cup \mathfrak{m}$.
3. $1 + \mathfrak{m} \subset A^\times$.

Exercise 18.
Let A be a ring and S a multiplicative set in A.

1. Show that $S^{-1}A$ is a ring and $\alpha : A \to S^{-1}A$, $a \mapsto \frac{a}{1}$, a ring homomorphism.
2. Show that α satisfies the following universal property: for every ring homomorphism $f : A \to B$ such that $f(S) \subset B^\times$, there is a unique ring homomorphism $f_S : S^{-1}A \to B$ such that $f = f_S \circ \alpha$.
3. Show that $\mathfrak{p} \mapsto \langle \alpha(\mathfrak{p}) \rangle_{S^{-1}A}$ defines a bijection
$$\{ \text{ prime ideals } \mathfrak{p} \text{ of } A \text{ with } \mathfrak{p} \cap S = \emptyset \} \quad \longrightarrow \quad \{ \text{ prime ideals of } S^{-1}A \}$$
4. Let M be an A-module. Show that $S^{-1}M$ is an $S^{-1}A$-module and that the map $M \to S^{-1}M$, $m \mapsto \frac{m}{1}$, is A-linear.

Exercise 19.
Let A be a ring and $a, b \in A$. We say that a *divides* b and write $a|b$ if $b = ac$ for some $c \in A$. An element $a \in A$ is a *prime* if $a \notin A^\times \cup \{0\}$ and if $a|bc$ implies $a|b$ or $a|c$ for any $b, c \in A$. An element $a \in A$ is *irreducible* if $a \notin A^\times \cup \{0\}$ and if $a = bc$ implies that either $b \in A^\times$ or $c \in A^\times$.

1. Show that an element $a \neq 0$ is prime if and only if (a) is a prime ideal.
2. Let A be an integral domain. Show that every prime element in A is irreducible.

Exercise 20.
Let $a, b \in \mathbb{Z}$. Denote by $\gcd(a, b)$ the greatest common divisor and $\operatorname{lcm}(a, b)$ the least common multiple of a and b. Show that
$$(a) + (b) = \big(\gcd(a, b)\big), \qquad (a) \cap (b) = \big(\operatorname{lcm}(a, b)\big) \qquad \text{and} \qquad (a) \cdot (b) = (ab).$$

Exercise 21.
Formulate and prove the three isomorphism theorems for rings.

Exercise 22.
Recall the statements and proofs of the principal divisor theorem and the classification of finitely generated modules over a principal ideal domain.