Exercises for Commutative Algebra
List 1 to hand in at 17.3.2017

Exercise 1.
Let A be a ring. Show that in the categories Sets and $A-\text{Mod}$ a morphism, which is an (A-linear) map, is

1. a monomorphism if and only if it is injective;
2. an epimorphism if and only if it is surjective;
3. an isomorphism if and only if it is bijective.

Show that 1 and 3 also hold in the category Rings of rings and that surjective homomorphisms of rings are epimorphisms. Show that localizations of rings at multiplicative subsets are epimorphisms. Conclude that in Rings not every epimorphism is surjective and that there are ring homomorphisms that are *not* isomorphisms, but both monomorphisms and epimorphisms.

Exercise 2.
Let A be a ring. Show that the category $A-\text{Mod}$ contains a terminal and an initial object, all equalizers and coequalizers, and all products and coproducts (for arbitrary families $\{M_i\}_{i\in I}$ of A-modules).

Exercise 3.
Show that the category of fields and field homomorphisms does neither contain terminal objects, nor initial objects, nor products, nor coproducts, nor coequalizers in general.

Exercise 4.
Let $f, g : A \to B$ be two morphisms of a category $\mathcal{C}$.

1. Show that the canonical projection $\pi_A : \text{eq}(f, g) \to A$ is a monomorphism.
2. Show that the canonical injection $\iota_B : B \to \text{coeq}(f, g)$ is an epimorphism.