Exercises for Commutative Algebra
List 3 to hand in at 31.3.2017

Exercise 1.
Verify the following isomorphisms:

$$A/I \otimes_A M \simeq M/IM, \qquad S^{-1}A \otimes_A M \simeq S^{-1}M, \qquad \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$$

where A is a ring, $I \subset A$ is an ideal, M is an A-module and $S \subset A$ is a multiplicative subset, and n, m are positive integers with greatest common divisor d.

Exercise 2.
Let A be a local ring and M and N A-modules such that $M \otimes_A N = 0$. Show that either $M = 0$ or $N = 0$. Is this conclusion true if A is not local?

Hint: Apply Nakayama's Lemma to reduce the question to the case of a field by dividing by the maximal ideal.

Exercise 3.
Let A be a ring and P an A-module. Show that the following are equivalent.

1. $\mathrm{Hom}_A(P, -)$ is an exact functor.
2. There is an A-module Q such that $P \oplus Q$ is free.
3. For every surjective homomorphism $f : N \to M$ of A-modules and every homomorphism $g : P \to M$ there exists a homomorphism $h : P \to N$ such that $g = f \circ h$.

An A-module with these properties is called *projective*. Conclude that every free A-module is projective and every projective A-module is flat.

Exercise 4.
Let $f : A \to B$ be a homomorphism of rings and N a B-module. Let f^*N be the A-module given by restriction of scalars. Let $s : N \to f^*N \otimes_A B$ be the A-linear map with $n \mapsto n \otimes 1$ and $p : f^*N \otimes_A B \to N$ the A-linear map with $n \otimes b \mapsto b.n$. Show that $p \circ s$ is the identity on N and conclude that N is a direct summand of $f^*N \otimes_A B$.